

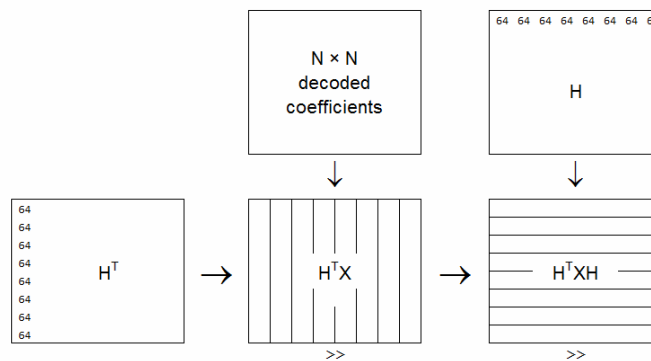
An algorithm with reduced number of multiplications for computing the 1-D integer transform of HEVC using recursive factorization and odd-part circulant transform

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Abstract – The HEVC Standard [3] defines an integer transform coding process together with a unified integer transform matrix to approximate the one-dimensional DCT for all transform sizes. This paper presents an algorithm that reduces the number of multiplications in the decoder for computing these matrix-vector products by finding symmetries in the integer DCT matrix and by transforming the odd DCT parts with permutation matrices to a circulant form, which gives possibilities for faster computation through recursive decomposition. The overall result in the case of the largest 32-point 1-D transform with this method is 120 multiplications (88% decrease) – without counting several constants, which can be implemented by shift. The algorithm is systematic and can be applied to any order- $2n$ real or integer transform.

Introduction

H.265/HEVC utilizes discrete cosine transform (DCT) coding for variable sizes of $N \times N$ image blocks ($N=2^n$, $2 \leq n \leq 5$). The two dimensional transform is computed as $2 \times N$ one dimensional transform first on columns then rows of the input block in the form of integer matrix-vector multiplication, results are shifted right with rounding after each 1-D transform.



Principles of the 2-D inverse integer transform

One 32×32 integer matrix is defined by the Standard for the 1-D transform; lower order- N transforms use a subset of the same matrix values. The integer transform process ensures bit-exact results between encoders and decoders and is a good approximation of the real DCT transform.

The naïve matrix-vector product on the largest 32-point transform would require 1024 and for the whole 32×32 block the two dimensional transform would require 65536 multiplications. Finding fast algorithms for computing these high order integer transforms could greatly speed up the decoding

process. The main focus of this paper is the inverse transform for the HEVC decoder – but it is worth mentioning that the algorithm for the forward transform has exactly the same complexity.

In Part I the number of multiplications required to solve the matrix-vector product are recursively reduced based on symmetries of the integer matrix and has been derived solely by observing the matrix values. The explanation lies in the relationship with the real DCT matrix reflecting the same alternating even/odd symmetric structure. It leads to an algorithm similar to other recursive DCT kernel factorizations known from the literature, yet it is discussed in detail as it gives the basic structure for the final algorithm. With this method the 32-point transform can be computed with 340 multiplications (66% decrease).

Part II focuses on reducing the number of multiplications for computing the remaining odd DCT matrix-vector products. With a rather systematic approach I have found a class of permutation matrices, which transforms the odd DCT matrices (A) to a sort of circulant form (C) by the equation $C = PAP^{-1}$, where P is a signed permutation matrix. Then a procedure that recursively calculate the solution with 9, 27 and 81 parallel multiplications for the odd part of the 8×8 , 16×16 and 32×32 DCT matrix, respectively. Integrating these findings into the fully factorized algorithm of Part I the final algorithm further reduces the number of multiplications to 3, 12, 39 and 120 for the 4-, 8-, 16- and 32-point 1-D transform of HEVC.

In Part III code for the inverse transform (HEVC decoder) will be implemented in C for all transform sizes.

Part I. Recursive reduction of multiplications based on symmetries

Computing the 1-D inverse integer transform in the decoder is equivalent to the matrix-vector multiplication of $y_n = H_n x_n$, where H is the integer matrix defined by the Standard, x and y are column vectors with $n = 4, 8, 16$ or 32 elements. Lower transform sizes sub-sample H , denoted by shaded values:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
64	90	90	90	89	88	87	85	83	82	80	78	75	73	70	67	64	61	57	54	50	46	43	38	36	31	25	22	18	13	9	4
64	90	87	82	75	67	57	46	36	22	9	-4	-18	-31	-43	-54	-64	-73	-80	-85	-89	-90	-90	-88	-83	-78	-70	-61	-50	-38	-25	-13
64	88	80	67	50	31	9	-13	-36	-54	-70	-82	-89	-90	-87	-78	-64	-46	-25	-4	18	38	57	73	83	90	90	85	75	61	43	22
64	85	70	46	18	-13	-43	-67	-83	-90	-87	-73	-50	-22	9	38	64	82	90	88	75	54	25	-4	-36	-61	-80	-90	-89	-78	-57	-31
64	82	57	22	-18	-54	-80	-90	-83	-61	-25	13	50	78	90	85	64	31	-9	-46	-75	-90	-87	-67	-36	4	43	73	89	88	70	38
64	78	43	-4	-50	-82	-90	-73	-36	13	57	85	89	67	25	-22	-64	-88	-87	-61	-18	31	70	90	83	54	9	-38	-75	-90	-80	-46
64	73	25	-31	-75	-90	-70	-22	36	78	90	67	18	-38	-80	-90	-64	-13	43	82	89	61	9	-46	-83	-88	-57	-4	50	85	87	54
64	67	9	-54	-89	-78	-25	38	83	85	43	-22	-75	-90	-57	4	64	90	70	13	-50	-88	-80	-31	-36	82	87	46	-18	-73	-90	-61
64	61	-9	-73	-89	-46	25	82	83	31	-43	-88	-75	-13	57	90	64	-4	-70	-90	-50	22	80	85	36	-38	-87	-78	-18	54	90	67
64	54	-25	-85	-75	-4	70	88	36	-46	-90	-61	18	82	80	13	-64	-90	-43	38	89	67	-9	-78	-83	-22	57	90	50	-31	-87	-73
64	46	-43	-90	-50	38	90	54	-36	-90	-57	31	89	61	-25	-88	-64	22	87	67	-18	-85	-70	13	83	73	-9	-82	-75	4	80	78
64	38	-57	-88	-18	73	80	-4	-83	-67	25	90	50	-46	-90	-31	64	85	9	-78	-75	13	87	61	-36	-90	-43	54	89	22	-70	-82
64	31	-70	-78	18	90	43	-61	-83	4	87	54	-50	-88	-9	82	64	-38	-90	-22	75	73	-25	-90	-36	67	80	-13	-89	-46	57	85
64	22	-80	-61	50	85	-9	-90	-36	73	70	-38	-89	-4	87	46	-64	-78	25	90	18	-82	-57	54	83	-13	-90	-31	75	67	-43	-88
64	13	-87	-38	75	61	-57	-78	36	88	-9	-90	-18	85	43	-73	-64	54	80	-31	-89	4	90	22	-83	-46	70	67	-50	-82	25	90
64	4	-90	-13	89	22	-87	-31	83	38	-80	-46	75	54	-70	-61	64	67	-57	-73	50	78	-43	-82	36	85	-25	-88	18	90	-9	-90
64	-4	-90	13	89	-22	-87	31	83	-38	-80	46	75	-54	-70	61	64	-67	-57	73	50	-78	-43	82	36	-85	-25	88	18	90	-9	90
64	-13	-87	38	75	-61	-57	78	36	-88	-9	90	-18	-85	43	73	-64	-54	80	31	-89	-4	90	-22	-83	46	70	-67	-50	82	25	-90
64	-22	-80	61	50	-85	-9	90	-36	-73	70	38	-89	4	87	-46	-64	78	25	-90	18	82	-57	-54	83	13	-90	31	75	-67	-43	88
64	-31	-70	78	18	-90	43	61	-83	-4	87	-54	-50	88	-9	-82	64	38	-90	22	75	-73	-25	90	-36	-67	80	13	-89	46	57	-85
64	-38	-57	88	-18	-73	80	4	-83	67	25	-90	50	46	-90	31	64	-85	9	78	-75	-13	87	-61	-36	90	-43	-54	89	-22	-70	82
64	-46	-43	90	-50	-38	90	-54	-36	90	-57	-31	89	-61	-25	88	-64	-22	87	-67	-18	85	-70	-13	83	-73	-9	82	-75	4	80	-78
64	-54	-25	85	-75	4	70	-88	36	46	-90	61	18	-82	80	-13	-64	90	-43	-38	89	-67	-9	78	-83	22	57	-90	50	31	-87	73
64	-61	-9	73	-89	46	25	-82	83	-31	-43	88	-75	13	57	-90	64	4	-70	90	-50	-22	80	-85	36	38	-87	78	-18	-54	90	-67
64	-67	9	54	-89	78	-25	-38	83	-85	43	22	-75	90	-57	-4	64	-90	70	-13	-50	88	-80	31	36	-82	87	-46	-18	73	-90	61
64	-73	25	31	-75	90	-70	22	36	-78	90	-67	18	38	-80	90	-64	13	43	-82	89	-61	9	46	-83	88	-57	4	50	-85	87	-54
64	-78	43	4	-50	82	-90	73	-36	-13	57	-85	89	-67	25	22	-64	88	-87	61	-18	-31	70	-90	83	-54	9	38	-75	90	80	46
64	-82	57	-22	-18	54	-80	90	-83	61	-25	-13	50	-78	90	-85	64	-31	-9	46	-75	90	-87	67	-36	-4	43	-73	89	-88	70	-38
64	-85	70	-46	18	13	-43	67	-83	90	-87	73	-50	22	9	-38	64	-82	90	-88	75	-54	25	4	-36	61	-80	90	-89	78	-57	31
64	-88	80	-67	50	-31	9	13	-36	54	-70	82	-89	90	-87	78	-64	46	-25	4	18	-38	57	-73	83	-90	90	-85	75	-61	43	-22
64	-90	87	-82	75	-67	57	-46	36	-22	9	4	-18	31	-43	54	-64	73	-80	85	-89	90	-90	88	-83	78	-70	61	-50	38	-25	13
64	-90	90	-90	89	-88	87	-85	83	-82	80	-78	75	-73	70	-67	64	-61	57	-54	50	-46	43	-38	36	-31	25	-22	18	-13	9	4

H_{32}

First let us examine H_8 , the 8x8 integer matrix used for the 8-point inverse transform then extend the procedure for higher order-N. Constants of $K_{i,j}$ in even columns appear symmetric, while in odd columns anti-symmetric:

	0	1	2	3	4	5	6	7
0	64	89	83	75	64	50	36	18
1	64	75	36	-18	-64	-89	-83	-50
2	64	50	-36	-89	-64	18	83	75
3	64	18	-83	-50	64	75	-36	-89
4	64	-18	-83	50	64	-75	-36	89
5	64	-50	-36	89	-64	-18	83	-75
6	64	-75	36	18	-64	89	-83	50
7	64	-89	83	-75	64	-50	36	-18

H_8

$$\begin{cases} K_{i,j} = K_{N-1-i,j}, & j \text{ even} \\ K_{i,j} = -K_{N-1-i,j}, & j \text{ odd} \end{cases}$$

$$N = 8, 0 \leq i \leq N/2 - 1, 0 \leq j \leq N - 1,$$

Rearranging columns to even- then odd order yields M_8 , revealing two sub-matrices of H_4 and A_4 appearing in reversed row order in the bottom half, denoted by r . H_4 is the 4x4 integer transform matrix itself, while A_4 is the odd part of the order-8 integer transform matrix:

	0	2	4	6	1	3	5	7
64	83	64	36	89	75	50	18	
64	36	-64	-83	75	-18	-89	-50	
64	-36	-64	83	50	-89	18	75	
64	-83	64	-36	18	-50	75	-89	
64	-83	64	-36	-18	50	-75	89	
64	-36	-64	83	-50	89	-18	-75	
64	36	-64	-83	-75	18	89	50	
64	83	64	36	-89	-75	-50	-18	

M_8

H_4	A_4
H_4^r	$-A_4^r$

h_0	a_0
h_1	a_1
h_2	a_2
h_3	a_3
h_3	$-a_3$
h_2	$-a_2$
h_1	$-a_1$
h_0	$-a_0$

Instead of solving H_8x with 64 multiplications, the same result can be achieved with 16+16=32 multiplications by solving the two sub-matrix-vector products of H_4x_{even} and A_4x_{odd} , followed by addition for the first half of y and subtraction (in reverse order) for the second half*. In matrix form:

$$y = H_8x = M_8 \begin{pmatrix} x_{even} \\ x_{odd} \end{pmatrix} = B_8 \begin{pmatrix} H_4 & \\ & A_4 \end{pmatrix} \begin{pmatrix} x_{even} \\ x_{odd} \end{pmatrix},$$

where B_8 is a symmetric matrix for addition and subtraction of the half-vector terms $h_i x_{even}$ and $a_i x_{odd}$ in proper order. B_8 can also be partitioned into 4 identity- and reverse identity sub-matrices:

$$B_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_4 & I_4 \\ I_4^r & -I_4^r \end{bmatrix}$$

Furthermore, H_4 above shows the same type of alternating even- and odd column symmetries and after similar column arrangement yields M_4 , with two sub-matrices of H_2 and A_2 :

* Reversing the order of the lower half can be either implemented into x -permutation or the order of subtractions in B .

	0	1	2	3		0	2	1	3		0	2	1	3
0	64	83	64	36		64	64	83	36		H ₂		A ₂	
1	64	36	-64	-83		64	-64	36	-83					
2	64	-36	-64	83		64	-64	-36	83				-A ₂ ^r	
3	64	-83	64	-36		64	64	-83	-36					
	H ₄					M ₄								

Suggesting H_4x (and the 4-point transform) can be solved with 4+4=8 multiplications instead of 16 according to

$$y = H_4x = M_4 \begin{pmatrix} x_{even} \\ x_{odd} \end{pmatrix} = B_4 \begin{pmatrix} H_2 & \\ & A_2 \end{pmatrix} \begin{pmatrix} x_{even} \\ x_{odd} \end{pmatrix} = \begin{pmatrix} I_2 & I_2 \\ I_2^r & -I_2^r \end{pmatrix} \begin{pmatrix} H_2 & \\ & A_2 \end{pmatrix} \begin{pmatrix} x_{even} \\ x_{odd} \end{pmatrix}$$

Furthermore, the upper 2x2 block of M_4 is in the same symmetry structure, the 2-point transform matrix, and thus can be computed with 2 multiplications:

$$y = \begin{pmatrix} 64 & 64 \\ 64 & -64 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = B_2 \begin{pmatrix} 64 & \\ & 64 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 64 & \\ & 64 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

As a result, multiplications were halved, and then recursively halved in each even half until scalar operations have been reached. H_8 has been factorized into a block diagonalized (A) and an additive matrix B, where $B=B_8B_4B_2^*$:

$$y = H_8x = BAPx = B \begin{pmatrix} A_1 & & & \\ & A_1 & & \\ & & A_2 & \\ & & & A_1 \end{pmatrix} Px = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 64 & & & \\ & 64 & & \\ & & 83 & 36 \\ & & 36 & -83 \\ & & & & 89 & 75 & 50 & 18 \\ & & & & 75 & -18 & -89 & -50 \\ & & & & 50 & -89 & 18 & 75 \\ & & & & 18 & -50 & 75 & -89 \end{pmatrix} \begin{pmatrix} x_0 \\ x_4 \\ x_2 \\ x_6 \\ x_1 \\ x_3 \\ x_5 \\ x_7 \end{pmatrix}$$

To verify the above equation through an example, below is the 8-point integer transform matrix with an arbitrary column vector x . The naïve matrix-vector multiplication of $y=H_8x$ uses 64 multiplications:

x	H ₈	y
165	64 89 83 75 64 50 36 18	7122
18	64 75 36 -18 -64 -89 -83 -50	23489
41	64 50 -36 -89 -64 18 83 75	10104
-25	64 18 -83 -50 64 75 -36 -89	8691
-75	64 -18 -83 50 64 -75 -36 89	4087
37	64 -50 -36 89 -64 -18 83 -75	-928
-112	64 -75 36 18 -64 89 -83 50	28775
23	64 -89 83 -75 64 -50 36 -18	3140

Entering matrices of A_n and B_n into a spreadsheet reveals the algorithm and the basis for code generation. The input vector is permuted to a recursive-even and odd order. First computing the 2-point transform with x_0 and x_4 : multiply by 64 (or left-shift) and add/sub results by B_2 . With x_2, x_6 perform matrix multiplication by A_2 . Add/sub results by B_4 . From x_{odd} perform matrix multiplication by A_4 . Add/sub results by B_8 . Data flows from left to right:

* More precisely B_4 and B_2 are extended to 8x8 by diagonal one-s before the matrix multiplication; while P is a permutation matrix.

	x		B₂		B₄		B₈		y															
x0	165	<table border="1"><tr><td>64</td></tr></table>	64	10560	<table border="1"><tr><td>1</td><td>1</td></tr></table>	1	1	5760	<table border="1"><tr><td>1</td><td>0</td><td>1</td><td>0</td></tr></table>	1	0	1	0	5131	<table border="1"><tr><td>1</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td></tr></table>	1	0	0	0	1	0	0	0	7122
64																								
1	1																							
1	0	1	0																					
1	0	0	0	1	0	0	0																	
x4	-75	<table border="1"><tr><td>64</td></tr></table>	64	-4800	<table border="1"><tr><td>1</td><td>-1</td></tr></table>	1	-1	15360	<table border="1"><tr><td>0</td><td>1</td><td>0</td><td>1</td></tr></table>	0	1	0	1	26132	<table border="1"><tr><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td></tr></table>	0	1	0	0	0	1	0	0	23489
64																								
1	-1																							
0	1	0	1																					
0	1	0	0	0	1	0	0																	
x2	41			<table border="1"><tr><td>83</td><td>36</td></tr></table>	83	36	-629	<table border="1"><tr><td>0</td><td>1</td><td>0</td><td>-1</td></tr></table>	0	1	0	-1	4588	<table border="1"><tr><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td></tr></table>	0	0	1	0	0	0	1	0	10104	
83	36																							
0	1	0	-1																					
0	0	1	0	0	0	1	0																	
x6	-112			<table border="1"><tr><td>36</td><td>-83</td></tr></table>	36	-83	10772	<table border="1"><tr><td>1</td><td>0</td><td>-1</td><td>0</td></tr></table>	1	0	-1	0	6389	<table border="1"><tr><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>1</td></tr></table>	0	0	0	1	0	0	0	1	8691	
36	-83																							
1	0	-1	0																					
0	0	0	1	0	0	0	1																	
x1	18					<table border="1"><tr><td>89</td><td>75</td><td>50</td><td>18</td></tr></table>	89	75	50	18	1991	<table border="1"><tr><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>-1</td></tr></table>	0	0	0	1	0	0	0	-1	4087			
89	75	50	18																					
0	0	0	1	0	0	0	-1																	
x3	-25					<table border="1"><tr><td>75</td><td>-18</td><td>-89</td><td>-50</td></tr></table>	75	-18	-89	-50	-2643	<table border="1"><tr><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>-1</td><td>0</td></tr></table>	0	0	1	0	0	0	-1	0	-928			
75	-18	-89	-50																					
0	0	1	0	0	0	-1	0																	
x5	37					<table border="1"><tr><td>50</td><td>-89</td><td>18</td><td>75</td></tr></table>	50	-89	18	75	5516	<table border="1"><tr><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>-1</td><td>0</td><td>0</td></tr></table>	0	1	0	0	0	-1	0	0	28775			
50	-89	18	75																					
0	1	0	0	0	-1	0	0																	
x7	23					<table border="1"><tr><td>18</td><td>-50</td><td>75</td><td>-89</td></tr></table>	18	-50	75	-89	2302	<table border="1"><tr><td>1</td><td>0</td><td>0</td><td>0</td><td>-1</td><td>0</td><td>0</td><td>0</td></tr></table>	1	0	0	0	-1	0	0	0	3140			
18	-50	75	-89																					
1	0	0	0	-1	0	0	0																	
			A₂		A₄																			

This algorithm uses 22 – or 20 not counting by 64 – multiplications with equal result computed as a series of matrix-vector products of $B_n x$ and $A_n x$, where only $A_n x$ requires multiplications (in Part II we will improve on computing these). The input vector is permuted, B_n are symmetric additive- and A_n are the integer odd part DCT matrices.

By observing H_{32} , the whole matrix appears in a similar structure showing alternating even/odd column symmetries. Rearranging columns* of H_{32} starting from the lowest transform size reveals the four order-N matrices of M_4 , M_8 , M_{16} and M_{32} used for different transform sizes:

0	16	8	24	4	12	20	28	2	6	10	14	18	22	26	30	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31		
64	64	83	36	89	75	50	18	90	87	80	70	57	43	25	9	90	90	88	85	82	78	73	67	61	54	46	38	31	22	13	4		
64	-64	-36	-83	75	-18	-89	-50	87	57	9	-43	-80	-90	-70	-25	90	82	67	46	22	-4	-31	-54	-73	-85	-90	-88	-78	-61	-38	-13		
64	-64	-36	83	50	-89	-18	75	80	9	-70	-87	-25	67	90	43	88	67	31	-13	-54	-82	-90	-78	-46	-4	38	73	90	85	61	22		
64	64	-83	-36	18	-50	75	-89	70	-43	-87	9	90	25	-80	-57	85	46	-13	-67	-90	-73	-22	38	82	88	54	-4	-61	-90	-78	-31		
64	64	-83	-36	-18	50	-75	89	57	-80	-25	90	-9	-87	43	70	82	22	-54	-90	-61	13	78	85	31	-46	-90	-67	4	73	88	38		
64	-64	36	-83	-75	18	89	50	25	-70	90	-80	43	9	-57	87	73	-31	-90	-22	78	67	-38	-90	-13	82	61	-46	-88	-4	85	54		
64	64	83	36	-89	-75	-50	-18	9	-25	43	-57	70	-80	87	-90	67	-54	-78	38	85	-22	-90	4	90	13	-88	-31	82	46	-73	-61		
64	64	83	36	-89	-75	-50	-18	-9	25	-43	57	-70	80	-87	90	61	-73	-46	82	31	-88	-13	90	-4	-90	22	85	-38	-78	54	67		
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64	64	-83	-36	-18	50	-75	89	-57	80	25	-90	9	87	-43	70	38	-88	73	-4	-67	90	-46	-31	85	-78	13	61	-90	54	22	-82		
64	64	-83	-36	-18	50	-75	89	-70	43	87	-9	-90	-25	80	57	31	-78	90	-61	4	54	-88	82	-38	-22	73	-90	67	-13	-46	85		
64	-64	36	-83	-75	18	89	50	-9	70	87	25	-57	-90	-43	22	-61	85	-90	73	-38	-4	46	-78	90	-82	54	-13	-31	67	-88			
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64	64	83	36	89	75	50	18	-90	-87	-80	-70	-57	-43	-25	-9	4	-13	22	-31	38	-46	54	-61	67	-73	78	-82	85	-88	90	-90		
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64	-64	-36	83	-50	89	-18	75	-80	-9	70	87	25	-57	-90	-43	-22	61	-85	90	-73	38	4	46	78	-90	82	-54	13	31	-67	88		
64	64	-83	-36	18	-50	75	-89	-70	43	87	-9	-90	-25	80	57	-31	78	90	61	-4	54	88	82	-38	-22	73	-90	67	-13	-46	85		
64	64	-83	-36	-18	50	-75	89	-57	80	25	-90	9	87	-43	70	-38	88	-73	4	67	-90	46	31	-85	78	-13	-61	90	-54	-22	82		
64	-64	36	-83	-75	18	89	50	-18	-75	-43	90	-57	-25	87	-70	-9	80	-46	90	-38	-54	90	-31	-61	88	-22	-67	85	-13	-73	82	-4	-78
64	-64	36	-83	-75	18	89	50	-25	70	-90	80	-43	-9	57	-87	-54	85	4	-88	46	61	-82	-13	90	-38	-67	78	22	-90	31	73		
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64	-64	-36	83	-50	89	-18	-75	-43	-90	57	25	-87	70	9	-80	-78	4	82	73	-13	-85	-67	22	88	61	-31	-90	-54	38	90	46		
64	64	-83	-36	-18	50	-75	89	57	-80	-25	90	-9	-87	43	70	-82	-22	54	90	61	-13	-78	85	-31	46	90	67	-4	-73	-88	-38		
64	64	-83	-36	-18	50	-75	89	-70	43	-87	9	90	25	-80	-57	-85	-46	13	67	90	73	22	-38	-82	-88	-54	4	61	90	78	31		
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64	-64	36	-83	-75	-18	-89	-50	87	57	9	-43	-80	-90	-70	-25	-90	-82	-67	-46	-22	4	31	54	73	85	90	88	78	61	38	13		
64	64	83	36	89	75	50	18	90	87	80	70	57	43	25	9	-90	-90	-88	-85	-82	-78	-73	-67	-61	-54	-46	-38	-31	-22	-13	-4		

$$M_4 = \begin{pmatrix} H_2 & A_2 \\ H_2^r & -A_2^r \end{pmatrix}$$

$$M_8 = \begin{pmatrix} M_4 & A_4 \\ M_4^r & -A_4^r \end{pmatrix}$$

$$M_{16} = \begin{pmatrix} M_8 & A_8 \\ M_8^r & -A_8^r \end{pmatrix}$$

$$M_{32} = \begin{pmatrix} M_{16} & A_{16} \\ M_{16}^r & -A_{16}^r \end{pmatrix}$$

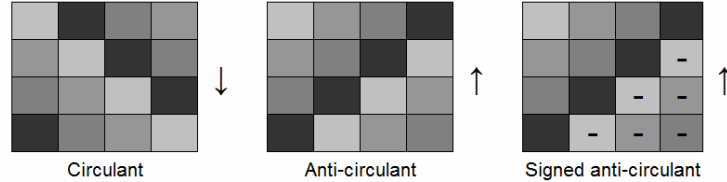
This already suggests some recursive computation as each M_n contains lower transform matrices. For higher order transforms, the 8x8 transform matrix is the upper left 8x8 block of the 16x16 transform matrix, and the 16x16 transform matrix is the upper left 16x16 block of the 32x32 transform matrix.

To verify whether this assumption holds the data flow is extended below with the order-16 and order-32 odd DCT matrices (A_8 and A_{16}) and the corresponding B_{16} and B_{32} additive matrices. The data flow is based on the full matrix factorization of H_{32} as:

* Rearranging columns and the corresponding permutation of the input column vector x causes no additional computational complexity when computing the matrix-vector product.

only the solution was published, but [1] used a permutation matrix to transform the odd DCT matrix A to some circular form of C according to $C=PAP^{-1}$, where P is a signed permutation matrix. An attractive feature of circulant matrices is solving the matrix-vector product with fewer multiplications, i.e. all multiplications by A narrows down to fewer multiplications used by C .

So the question and the quest of this paper is whether such P_n signed permutation matrix exists for all of the order- 2^n DCT odd matrix parts, how can be found from $2^n n!$ possibilities, what is the algorithm for the data flow and how to generate the C code for the HEVC decoder.



An $n \times n$ circulant matrix can be constructed by cycling a column vector *down*, resulting in a square matrix with n elements arranged so that the matrix has same values along diagonals. Let us call matrices with same values along anti-diagonals as *anti-circulant*, derived by cycling the first column vector *up*. According to the results presented in this paper the odd DCT matrix part can be transformed to a so called *signed-anti-circulant* format by applying $C=PAP^{-1}$, where P are various signed permutation matrices. The structure of signed-anti-circulant is similar to anti-circulant, same values along anti-diagonals, except from the rotated element is negated.

Example

As an example, let us solve the matrix multiplication for $y=Ax$ for A_4 , the 4×4 odd part of the order-8 integer DCT matrix. This would require 16 multiplications:

$$\begin{array}{c} \mathbf{A} \\ \hline \begin{array}{cccc} 89 & 75 & 50 & 18 \\ 75 & -18 & -89 & -50 \\ 50 & -89 & 18 & 75 \\ 18 & -50 & 75 & -89 \end{array} \end{array} \begin{array}{c} \mathbf{x} \\ \hline \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \end{array} = \begin{array}{c} \mathbf{y} \\ \hline \begin{array}{c} y_0 \\ y_1 \\ y_2 \\ y_3 \end{array} \end{array}$$

Let $C=PAP^{-1}$, where P is a permutation matrix:

$$\begin{array}{c} \mathbf{P} \\ \hline \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array} \end{array} \begin{array}{c} \mathbf{A} \\ \hline \begin{array}{cccc} 89 & 75 & 50 & 18 \\ 75 & -18 & -89 & -50 \\ 50 & -89 & 18 & 75 \\ 18 & -50 & 75 & -89 \end{array} \end{array} \begin{array}{c} \mathbf{P}^{-1} \\ \hline \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \end{array} = \begin{array}{c} \mathbf{C} \\ \hline \begin{array}{cc|cc} 89 & 50 & 18 & 75 \\ 50 & 18 & 75 & -89 \\ \hline 18 & 75 & -89 & -50 \\ 75 & -89 & -50 & -18 \end{array} \end{array}$$

C is not strictly a circulant matrix by definition, but has a circular structure: same values along anti-diagonals (i.e. *signed anti-circulant*). From the above $A= P^{-1}CP$ using that permutation matrices are orthogonal and $P^{-1}P=I$.

Substituting into the original equation we get $y = Ax = P^{-1}CPx$ or $y = P^{-1} (C (Px))$ based on matrix product associativity. Because of multiplying by P or P^{-1} is only row- or column permutation with sign-change, this essentially means that multiplications will only appear during computing the $(C)(permuted-x)$ product:

$$\begin{array}{|c|c|c|c|} \hline \mathbf{C} & & & \\ \hline 89 & 50 & 18 & 75 \\ \hline 50 & 18 & 75 & -89 \\ \hline 18 & 75 & -89 & -50 \\ \hline 75 & -89 & -50 & -18 \\ \hline \end{array} \quad \begin{array}{|c|} \hline \mathbf{Px} \\ \hline x_0 \\ \hline x_2 \\ \hline x_3 \\ \hline x_1 \\ \hline \end{array} = \begin{array}{|c|} \hline \mathbf{CPx} \\ \hline y_0 \\ \hline y_2 \\ \hline y_3 \\ \hline y_1 \\ \hline \end{array}$$

Now $C(Px)$ can be solved with fewer multiplications than the naïve 16. Each of the 2×2 blocks has two equal anti-diagonal values $\begin{bmatrix} u & d \\ d & v \end{bmatrix}$. Furthermore, partitioning C into 2×2 blocks we get

$$C = \begin{bmatrix} U & D \\ D & V \end{bmatrix}, \text{ where the two } D \text{ sub-matrices are again equal. Both the scalar and the matrix form of such structures can be solved with 3 multiplications instead of 4. For the scalar case of } \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} u & d \\ d & v \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \text{ we get:}$$

$$\begin{aligned} t &= d(x_0 + x_1) \\ y_0 &= t + (u - d)x_0 \\ y_1 &= t + (v - d)x_1 \end{aligned}$$

The matrix form is $\begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix} = \begin{bmatrix} U & D \\ D & V \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \end{bmatrix}$, where X, Y are the upper and the lower half of the column vectors. Then again

$$\begin{aligned} T &= D(X_0 + X_1) = \begin{bmatrix} 18 & 75 \\ 75 & -89 \end{bmatrix} \begin{bmatrix} x_0 + x_2 \\ x_1 + x_3 \end{bmatrix} = \begin{bmatrix} t_0 \\ t_1 \end{bmatrix} \\ Y_0 &= T + (U - D)X_0 = \begin{bmatrix} t_0 \\ t_1 \end{bmatrix} + \begin{bmatrix} 71 & -25 \\ -25 & 107 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \\ Y_1 &= T + (V - D)X_1 = \begin{bmatrix} t_0 \\ t_1 \end{bmatrix} + \begin{bmatrix} -107 & -125 \\ -125 & 71 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} \end{aligned}$$

These 3 matrix-vector products are the scalar case above, thus $C(Px)$ can be solved using $3 \times 3 = 9$ multiplications. If P_8 and P_{16} exist as well, we predict solving A_8x and $A_{16}x$ with 27 and 81 multiplications instead of the naïve 64 and 256 – respectively – for the odd DCT matrix parts of HEVC.

Finding a permutation matrix

The P permutation matrix in the Example above is a solution for A_4 , that $C=PAP^{-1}$ has a circulant anti-diagonal structure. For higher order A_n finding P_n is not trivial. There are $n!$ possible permutation matrices of size $n \times n$. For signed permutation matrices each of these has 2^n possible variations with variable ± 1 elements, reaching $2^n n!$ to rather high numbers for $n=8$ and $n=16$. The task is to find a subset of these, which results in $C=PAP^{-1}$ in the desired structure.

Material and method

Without solid combinatory, matrix- and group theory, I used another method to find the right permutation matrix for each A_n of the integer DCT matrix. That is a spreadsheet and systematic approach, trial and error. First entering the steps of the equation $C=PAP^T$ into a spreadsheet starting

with an all-zero P matrix*. The PA product permutes rows of A , while AP^T permutes columns. Starting with $P_{1,1} = 1$, then continue with entering ± 1 values manually into P row-by-row observing the result in C , making sure anti-diagonals get the same value. The C matrix is colored by positive/negative to easily observe values along anti-diagonals. An example of the process for A_8 :

A	<table border="1"><tr><td>90</td><td>87</td><td>80</td><td>70</td><td>57</td><td>43</td><td>25</td><td>9</td></tr><tr><td>87</td><td>57</td><td>9</td><td>-43</td><td>-80</td><td>-90</td><td>-70</td><td>-25</td></tr><tr><td>80</td><td>9</td><td>-70</td><td>-87</td><td>-25</td><td>57</td><td>90</td><td>43</td></tr><tr><td>70</td><td>-43</td><td>-87</td><td>9</td><td>90</td><td>25</td><td>-80</td><td>-57</td></tr><tr><td>57</td><td>-80</td><td>-25</td><td>90</td><td>-9</td><td>-87</td><td>43</td><td>70</td></tr><tr><td>43</td><td>-90</td><td>57</td><td>25</td><td>-87</td><td>70</td><td>9</td><td>-80</td></tr><tr><td>25</td><td>-70</td><td>90</td><td>-80</td><td>43</td><td>9</td><td>-57</td><td>87</td></tr><tr><td>9</td><td>-25</td><td>43</td><td>-57</td><td>70</td><td>-80</td><td>87</td><td>-90</td></tr></table>	90	87	80	70	57	43	25	9	87	57	9	-43	-80	-90	-70	-25	80	9	-70	-87	-25	57	90	43	70	-43	-87	9	90	25	-80	-57	57	-80	-25	90	-9	-87	43	70	43	-90	57	25	-87	70	9	-80	25	-70	90	-80	43	9	-57	87	9	-25	43	-57	70	-80	87	-90	<table 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The above solution leads to a signed permutation matrix of P_8 , which transforms A_8 to a signed anti-circulant C_8 . This 8×8 matrix-vector product can be solved with 27 multiplications instead of 64 based on a recursive, circulant solver algorithm.

Similarly, for A_{16} with this method we can obtain C_{16} , which requires 81 multiplications to solve the 16×16 matrix-vector product instead of 256:

90	88	31	90	67	-78	82	54	4	22	85	-13	61	46	38	73
88	31	90	67	-78	82	54	4	22	85	-13	61	46	38	73	90
31	90	67	-78	82	54	4	22	85	-13	61	46	38	73	90	88
90	67	-78	82	54	4	22	85	-13	61	46	38	73	90	88	31
67	-78	82	54	4	22	85	-13	61	46	3					

Other permutation matrices

Although for the algorithm to work one signed P_n permutation matrix was enough to find with the spreadsheet method, some patterns started revealing during this manual process of building up various permutation matrices. It turned out that there exist several permutations that give solutions, while some do not. For example $P_{1,1} = 1$, then $P_{2,4} = 1$ does not lead to any solution for A_8 above and it *closes* the circle after 4 steps. To systematically determine all possible signed permutation matrices that the result of $C=PAP^{-1}$ is in a circular form were divided into two parts:

- First with the positive odd DCT matrix $|A_n|$, find all positive P permutation matrices from $n!$ possibilities that arranges elements of $|C|$ into the desired circular order. The results are several *positive base permutations* and their cyclic permutations.
- Then for A_n and for each positive base permutation, determine signs from 2^n possibilities. The results are several *signed base permutations* and their signed-cyclic rotations. These are the final solutions for the algorithm and basis for code generation obtained programmatically and by mathematical deduction.

Permutation vector and mapping

For convenience, in the rest of this paper a specific signed permutation matrix P is denoted by the p permutation vector. The permutation vector contains column indices for each row where the permutation matrix has a positive 1 value – and negative column indices where the permutation matrix has a negative 1 value. When solving $C(Px)$ this is also equivalent to mapping the input vector in proper order and sign before entering the matrix multiplication. The mapping (m) is simply the zero-based equivalent of p . For example:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix} \leftarrow \begin{pmatrix} 1 \\ -3 \\ 4 \\ -2 \end{pmatrix} \leftarrow p = (1, -3, 4, -2) \quad CPx = CP \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \xrightarrow{m=(0,-2,3,-1)} = C \begin{pmatrix} x_0 \\ -x_2 \\ x_3 \\ -x_1 \end{pmatrix}$$

a) Positive base permutation vectors

These results have been derived by the spreadsheet method with a positive $|A_n|$ matrix. A *base permutation vector* is a specific permutation vector starting with 1 (that is $P_{1,1} = 1$). The table shows all the positive base permutation vectors that $C=PAP^{-1}$ is in a circular form. There were 2 different base permutation vectors for 4×4 , 4 vectors for 8×8 and 8 vectors for the 16×16 $|A_n|$ odd DCT matrices:

$ p _{n=4}$				$ p _{n=8}$								$ p _{n=16}$															
1	2	3	4	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	4	3	1	2	5	3	8	7	4	6	1	2	5	14	9	7	13	6	16	15	12	3	8	10	4	11
1	3	4	2	1	3	4	2	8	6	5	7	1	3	13	2	8	6	5	10	16	14	4	15	9	11	12	7
				1	6	4	7	8	3	5	2	1	6	4	7	8	14	12	2	16	11	13	10	9	3	5	15
				1	7	5	6	8	2	4	3	1	7	12	11	9	15	4	14	16	10	5	6	8	2	13	3
								1	10	12	6	9	2	4	3	16	7	5	11	8	15	13	14				
								1	11	4	10	8	3	12	15	16	6	13	7	9	14	5	2				
								1	14	13	15	8	11	5	7	16	3	4	2	9	6	12	10				
								1	15	5	3	9	10	13	11	16	2	12	14	8	7	4	6				

Cyclic permutations of the positive base vector

Cycling rows of a positive P permutation matrix keeps the circular structure of the resulting $|C|$ matrix (empirical result). This is equivalent to cyclic permutations of the \mathbf{p} vector. Below is the result of all 4 cyclic permutations of positive $\mathbf{p} = (1, 2, 4, 3)$. Note that half of the resulting $|C|$ matrices are the same and that only half of the original matrix values can appear along the main anti-diagonal (here either 50 or 75):

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2	4	3	1																																																																	
4	3	1	2																																																																	
3	1	2	4																																																																	

This has a meaning that finding a specific base permutation matrix with $P_{1,1} = 1$, which is a solution, all cyclic permutations of the corresponding permutation vector are also solutions. Based on deduction the number of positive permutation vectors that arranges elements of $|A_n|$ into proper order are:

- $2 \times 4 = \mathbf{8}$ for $|A_4|$ out of $4! = 24$ possible permutation matrices
- $4 \times 8 = \mathbf{32}$ for $|A_8|$ out of $8! = 40320$ possible permutation matrices
- $8 \times 16 = \mathbf{128}$ for $|A_{16}|$ out of $16! \approx 2 \times 10^{13}$ possible permutation matrices

b) Signed base permutation vectors

Permuting an A_n odd DCT matrix with a positive permutation matrix according to $C = PAP^{-1}$ arranges elements of C in proper diagonal order but with wrong signed values (first column below). Changing signs in a particular P may lead to several correct results, as illustrated here with 4 signed base permutations found for $|\mathbf{p}| = (1, 2, 4, 3)$, out of $2^4 = 16$ possible variations:

A	$\begin{bmatrix} 89 & 75 & 50 & 18 \\ 75 & -18 & -89 & -50 \\ 50 & -89 & 18 & 75 \\ 18 & -50 & 75 & -89 \end{bmatrix}$	$\begin{bmatrix} 89 & 75 & 50 & 18 \\ 75 & -18 & -89 & -50 \\ 50 & -89 & 18 & 75 \\ 18 & -50 & 75 & -89 \end{bmatrix}$	$\begin{bmatrix} 89 & 75 & 50 & 18 \\ 75 & -18 & -89 & -50 \\ 50 & -89 & 18 & 75 \\ 18 & -50 & 75 & -89 \end{bmatrix}$	$\begin{bmatrix} 89 & 75 & 50 & 18 \\ 75 & -18 & -89 & -50 \\ 50 & -89 & 18 & 75 \\ 18 & -50 & 75 & -89 \end{bmatrix}$	$\begin{bmatrix} 89 & 75 & 50 & 18 \\ 75 & -18 & -89 & -50 \\ 50 & -89 & 18 & 75 \\ 18 & -50 & 75 & -89 \end{bmatrix}$
P	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
PA	$\begin{bmatrix} 89 & 75 & 50 & 18 \\ 75 & -18 & -89 & -50 \\ 18 & -50 & 75 & -89 \\ 50 & -89 & 18 & 75 \end{bmatrix}$	$\begin{bmatrix} 89 & 75 & 50 & 18 \\ 75 & -18 & -89 & -50 \\ -18 & 50 & -75 & 89 \\ 50 & -89 & 18 & 75 \end{bmatrix}$	$\begin{bmatrix} 89 & 75 & 50 & 18 \\ -75 & 18 & 89 & 50 \\ -18 & 50 & -75 & 89 \\ -50 & 89 & -18 & -75 \end{bmatrix}$	$\begin{bmatrix} -89 & -75 & -50 & -18 \\ -75 & 18 & 89 & 50 \\ 18 & -50 & 75 & -89 \\ -50 & 89 & -18 & -75 \end{bmatrix}$	$\begin{bmatrix} -89 & -75 & -50 & -18 \\ 75 & -18 & -89 & -50 \\ 18 & -50 & 75 & -89 \\ 50 & -89 & 18 & 75 \end{bmatrix}$
P^T	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
C	$\begin{bmatrix} 89 & 75 & 18 & 50 \\ 75 & -18 & -50 & -89 \\ 18 & -50 & -89 & 75 \\ 50 & -89 & 75 & 18 \end{bmatrix}$	$\begin{bmatrix} 89 & 75 & -18 & 50 \\ 75 & -18 & 50 & -89 \\ -18 & 50 & -89 & -75 \\ 50 & -89 & -75 & 18 \end{bmatrix}$	$\begin{bmatrix} 89 & -75 & -18 & -50 \\ -75 & -18 & -50 & -89 \\ -18 & -50 & -89 & 75 \\ -50 & -89 & 75 & 18 \end{bmatrix}$	$\begin{bmatrix} 89 & 75 & -18 & 50 \\ 75 & -18 & 50 & -89 \\ -18 & 50 & -89 & -75 \\ 50 & -89 & -75 & 18 \end{bmatrix}$	$\begin{bmatrix} 89 & -75 & -18 & -50 \\ -75 & -18 & -50 & -89 \\ -18 & -50 & -89 & 75 \\ -50 & -89 & 75 & 18 \end{bmatrix}$
p	$\begin{bmatrix} 1 & 2 & 4 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -4 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & -4 & -3 \end{bmatrix}$	$\begin{bmatrix} -1 & -2 & 4 & -3 \end{bmatrix}$	$\begin{bmatrix} -1 & 2 & 4 & 3 \end{bmatrix}$

There is a sign-symmetry among the solutions. Applying a specific P and $-P$ results in the same C matrix, meaning solutions for a signed p permutation vector come in pairs that are negatives of each other. The following table has been generated programmatically by verifying all 2^n possible signed combinations, whether $C=PA P^{-1}$ becomes anti-circulant, for each of the positive base vector $|p_n|$ from the previous table. Note that there were always 2 signed solutions per positive base vector found, regardless of n :

$p_{n=4}$	$p_{n=8}$	$p_{n=16}$
$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 1 & -2 & -4 & -3 \\ 1 & 3 & 4 & 2 \\ 1 & -3 & 4 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 5 & -3 & -8 & 7 & -4 & 6 \\ 1 & -2 & 5 & 3 & -8 & -7 & -4 & -6 \\ 1 & 3 & -4 & 2 & 8 & 6 & 5 & -7 \\ 1 & -3 & -4 & -2 & 8 & -6 & 5 & 7 \\ 1 & 6 & 4 & 7 & 8 & -3 & -5 & 2 \\ 1 & -6 & 4 & -7 & 8 & 3 & -5 & -2 \\ 1 & 7 & -5 & -6 & -8 & -2 & 4 & -3 \\ 1 & -7 & -5 & 6 & -8 & 2 & 4 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 5 & 14 & -9 & 7 & -13 & 6 & -16 & 15 & -12 & 3 & 8 & -10 & 4 & 11 \\ 1 & -2 & 5 & -14 & -9 & -7 & -13 & -6 & -16 & -15 & -12 & -3 & 8 & 10 & 4 & -11 \\ 1 & 3 & 13 & 2 & 8 & -6 & 5 & -10 & 16 & 14 & 4 & -15 & -9 & 11 & 12 & 7 \\ 1 & -3 & 13 & -2 & 8 & 6 & 5 & 10 & 16 & -14 & 4 & 15 & -9 & -11 & 12 & -7 \\ 1 & 6 & 4 & -7 & -8 & 14 & -12 & -2 & 16 & -11 & -13 & -10 & 9 & -3 & 5 & 15 \\ 1 & -6 & 4 & 7 & -8 & -14 & -12 & 2 & 16 & 11 & -13 & 10 & 9 & 3 & 5 & -15 \\ 1 & 7 & -12 & 11 & 9 & -15 & -4 & 14 & -16 & -10 & -5 & -6 & -8 & 2 & -13 & 3 \\ 1 & -7 & -12 & -11 & 9 & 15 & -4 & -14 & -16 & 10 & -5 & 6 & -8 & -2 & -13 & -3 \\ 1 & 10 & 12 & -6 & 9 & -2 & 4 & 3 & -16 & 7 & 5 & -11 & -8 & -15 & 13 & -14 \\ 1 & -10 & 12 & 6 & 9 & 2 & 4 & -3 & -16 & -7 & 5 & 11 & -8 & 15 & 13 & 14 \\ 1 & 11 & -4 & -10 & -8 & 3 & 12 & 15 & 16 & 6 & 13 & 7 & 9 & 14 & -5 & 2 \\ 1 & -11 & -4 & 10 & -8 & -3 & 12 & -15 & 16 & -6 & 13 & -7 & 9 & -14 & -5 & -2 \\ 1 & 14 & -13 & 15 & 8 & 11 & -5 & -7 & 16 & -3 & -4 & 2 & -9 & 6 & -12 & -10 \\ 1 & -14 & -13 & -15 & 8 & -11 & -5 & 7 & 16 & 3 & -4 & -2 & -9 & -6 & -12 & 10 \\ 1 & 15 & -5 & -3 & -9 & -10 & 13 & -11 & -16 & -2 & 12 & 14 & 8 & -7 & -4 & 6 \\ 1 & -15 & -5 & 3 & -9 & 10 & 13 & 11 & -16 & 2 & 12 & -14 & 8 & 7 & -4 & -6 \end{bmatrix}$

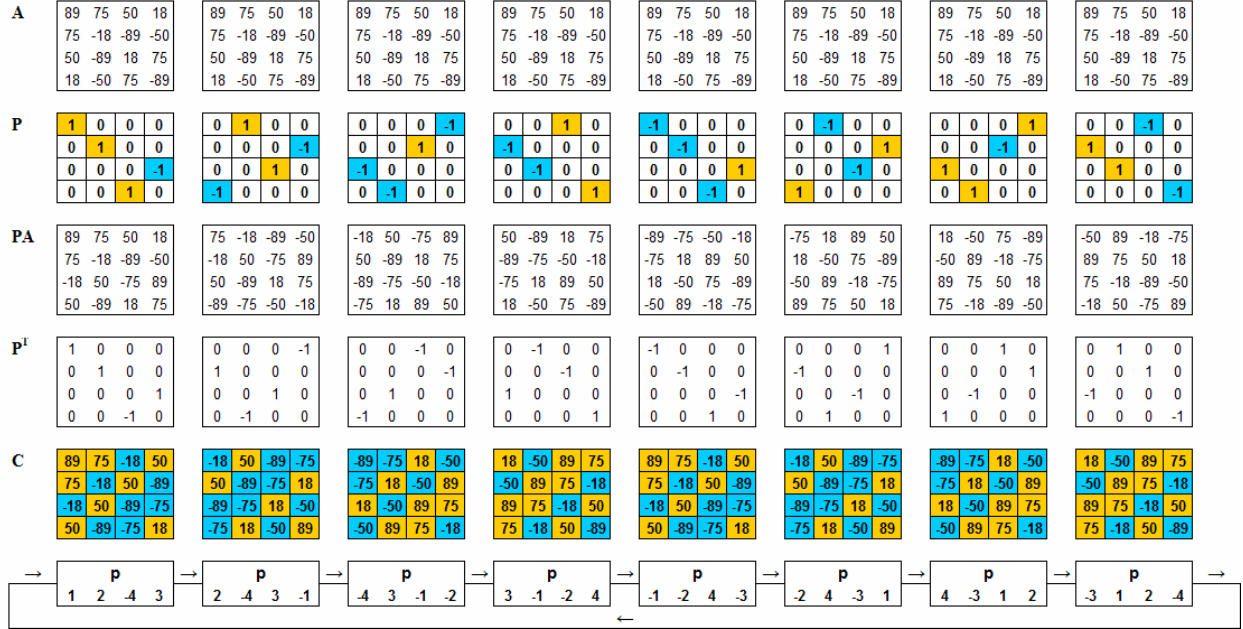
Any of these signed permutation vectors together with their negated pair are solutions for the final algorithm.

Cyclic permutations of the signed permutation vector

As for cycling rows of positive P , there is a similar rule when cycling a signed P permutation matrix. To keep $C=PA P^{-1}$ signed anti-circulant, the bottom row of P has to be negated, which is equivalent to

cycling p left then negating the last element (“sign-cycling”). An interesting correlation to how a sign-anti-circulant matrix is defined the same way, sign-cycling its columns.

For example sign-cycling $p = (1, 2, -4, 3)$ there are 8 different permutations. These appear as 4 negated pairs, where again, applying the negated vector results in the same C matrix:



Investigating other and also higher order permutation vectors we can deduce the following symmetry rule for the result of $C = PAP^T$ based on sign-cycling a specific order- N permutation vector $2 \times N$ -times:

No. of sign-cyclic permutations of p_n			
$\times 2N$			
$\times N$		$\times N$	
corresponding $p_{0..n}$		corresponding $-p_{0..n}$	
$\times N/2$	$\times N/2$	$\times N/2$	$\times N/2$
$C_{0..n/2}$	$-C_{0..n/2}$	$C_{0..n/2}$	$-C_{0..n/2}$

This finding has the importance for computing the integer multipliers for the circulant solver algorithm as the C_n matrix and all its sub-matrix products (D , U and V) will be different. There is enough to list and investigate constants for only $N/2$ sign-cyclic permutations of an order- N signed permutation vector.

For example generating the final 9 multipliers for all 8 sign-cyclic permutations of $p = (1, 2, -4, 3)$, for $N=4$ we get:

1	2	-4	3	82	25	46	-68	50	-139	54	-125	232	C_0
2	-4	3	-1	-54	125	-232	-14	-75	93	82	25	46	C_1
-4	3	-1	-2	-82	-25	-46	68	-50	139	-54	125	-232	$-C_0$
3	-1	-2	4	54	-125	232	14	75	-93	-82	-25	-46	$-C_1$
-1	-2	4	-3	82	25	46	-68	50	-139	54	-125	232	C_0
-2	4	-3	1	-54	125	-232	-14	-75	93	82	25	46	C_1
4	-3	1	2	-82	-25	-46	68	-50	139	-54	125	-232	$-C_0$
-3	1	2	-4	54	-125	232	14	75	-93	-82	-25	-46	$-C_1$

A compiled list of all multiplier constants for the order-4, -8 and -16 can be found in the Appendix for $N/2$ sign-cycled permutations each.

Summary of results

Now we can answer the original question in the beginning of Part II. Yes, there exist numerous signed permutation matrices that transforms the A_n odd-DCT matrix of any order- N to some circulant form by applying $C=PAP^{-1}$.

There were different numbers of positive base permutations for a particular n that arrange $|A_n|$ to anti-circulant form. From $n!$ possibilities these were derived manually by the spreadsheet method. For A_n , every positive base permutation has 2 signed base permutations, and every signed base permutation has $2 \times n$ signed-cyclic variations, which arrange A_n to the desired signed-anti-circulant form.

Multiplying C by a column vector requires $3^{\log_2 n}$ multiplications instead of n^2 based on the circulant solver algorithm. Follows a comparison of the first ten $n \times n$ such anti-circulant matrices and the number of required multiplications to solve the matrix-vector product with the naïve and recursive circulant multiplication method:

$\log_2 n$	1	2	3	4	5	6	7	8	9	10
n	2	4	8	16	32	64	128	256	512	1024
Naïve: n^2	4	16	64	256	1024	4096	16384	65536	262144	1048576
Circulant: $3^{\log_2 n}$	3	9	27	81	243	729	2187	6561	19683	59049
Decrease:	25 %	44 %	58 %	68 %	76 %	82 %	87 %	90 %	92 %	94 %

Which signed permutation vector is chosen for the final algorithm may have impact on the generated code. For permuting the input vector, one can prefer the least number of negations to run the code slightly faster. Below are a permutation with the least number of negations for each A_n :

1. $\mathbf{p}_4 = (1, 3, 4, 2)$ with zero for A_4
2. $\mathbf{p}_8 = (8, 7, 4, 6, 1, -2, 5, 3)$ with one for A_8
3. $\mathbf{p}_{16} = (3, 12, 15, 16, 6, 13, 7, 9, 14, -5, 2, -1, -11, 4, 10, 8)$ with three for A_{16}

Another consideration is the different multipliers used. Each permutation results in different U , D and V sub-matrices used to compute the constant multipliers. From the $2 \times N$ signed-cyclic variations $N/2$ will result in different multiplier constant sets. Multiplication-free implementations can chose a specific set, which is the best candidate for dyadic decomposition. Appendix includes the full compiled list of multipliers for each A_n odd DCT matrices.

The final algorithm

Based on the results in Part II now all four $A_n \times$ product can be solved with fewer multiplications by using the equation $P^T C P x$. For implementing the final algorithm, 3 signed permutation vectors were picked from the list, corresponding to P_4 , P_8 and P_{16} :

- $\mathbf{p}_4 = (1, 3, 4, 2)$
 $\mathbf{p}_8 = (8, 7, 4, 6, 1, -2, 5, 3)$
 $\mathbf{p}_{16} = (3, 12, 15, 16, 6, 13, 7, 9, 14, -5, 2, -1, -11, 4, 10, 8)$

Furthermore, P can be incorporated into the input vector mapping and P^T into the corresponding B_n matrices at no additional computational cost, making the algorithm even faster. For the *input mapping*, multiplying P_n with the corresponding $x_{n\text{-odd}}$ lists we get:

$$m_4 = (x_{4\text{even}}, x_{4\text{odd}}) = (0, 2, 1, 3),$$

$$m_8 = (2m_4, \mathbf{P}_4 x_{8\text{odd}}),$$

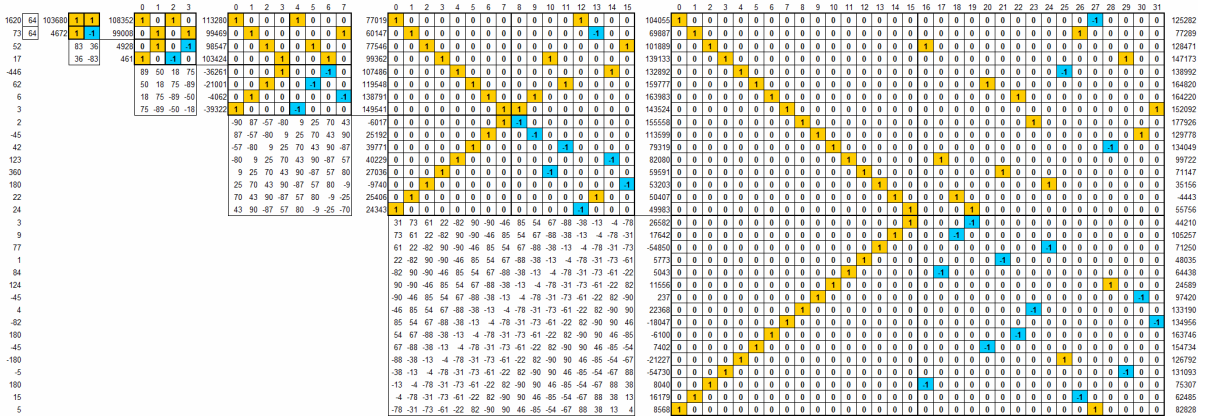
$$m_{16} = (2m_8, \mathbf{P}_8 x_{16\text{odd}}),$$

$$m_{32} = (2m_{16}, \mathbf{P}_{16} x_{32\text{odd}})$$

For the *output*, multiplying the left part of each B_n with $P_{n/2}^T$ we get:

$$B'_n = \begin{bmatrix} I_{n/2} & I_{n/2} P_{n/2}^T \\ I_{n/2}^r & -I_{n/2}^r P_{n/2}^T \end{bmatrix}$$

For illustration, the data flow of Part I has been modified. A_4 , A_8 and A_{16} are now the signed anti-circulant C_4 , C_8 and C_{16} matrices (B_4 remains the same, A_2 is already *circulant*). The 3 permutation matrices of P_4 , P_8 and P_{16} have been incorporated into the input vector mapping and P^T into the corresponding B_8 , B_{16} and B_{32} matrices:



This algorithm solves the HEVC N-point 1-D integer transform with the following number of multiplications:

- 4-point: 2-point (shift left – zero multiplication) + 3 multiplications for $A_2=C_2$: **3**
- 8-point: 4-point + 9 multiplications for C_4 : **12**
- 16-point: 8-point + 27 multiplications for C_8 : **39**
- 32-point: 16-point + 81 multiplications for C_{16} : **120**

The forward integer transform

It is worth mentioning that the forward integer transform can also be computed according to the final algorithm as a series of matrix-vector multiplications of $A_n x$ and $B_n x$ with the following modifications. The data flow is inverted: it starts with a series of additions/subtractions of the input column vector. Reflecting the transposed integer DCT matrix, symmetries appear for rows instead of columns; therefore the B_n matrices are also transposed with a common structure of

The circulant solver for the matrix-vector multiplication of Cx .

```
circulant_solver(C, X, N)
{
    if (N==1) multiply
    else
    {
        diag = circulant_solver(D, X0+X1, N/2)
        upper = circulant_solver(U, X0, N/2)
        lower = circulant_solver(L, X1, N/2)
        Y0 = diag + upper
        Y1 = diag + lower
    }
}
```

For the circulant solver the goal was to serialize this recursive adder/multiplier algorithm with a set of integer constants. Another challenge was to avoid of using temporarily variables. For a complete implementation based on 3 example signed permutation vectors see `transform.c`. These vectors were chosen with the most number of multipliers that can be implemented as shifts.

Performance measurements

A simple measurement has been made against an un-optimized, naïve matrix multiplication routine using MS VS2005, compared to an optimization disabled and optimized for speed (/O2) version of `transform.c`. Seven measurements have been performed and the averages of the results are presented in the table.

	Iterated x times	Naïve	Opt disabled		Optimized for speed	
		Duration (seconds)	Duration (seconds)	Speed increase	Duration (seconds)	Speed increase
32-point	5 million	18.56	2.51	7.39	0.77	23.98
16-point	20 million	19.77	3.52	5.61	1.20	16.45
8-point	50 million	12.50	3.18	3.93	1.14	10.96
4-point	160 million	12.10	3.67	3.29	0.94	12.85

AMD Phenom™ II X4 965 @ 3.40 GHz, 64-bit Windows™ 7.

Appendix: The circulant solver algorithm

A signed anti-circulant matrix is also recursively circulant, as illustrated by the sum-s of values of lower order matrices below with the following 8x8 matrix (C_0):

90	80	-70	87	9	43	57	-25
80	-70	87	9	43	57	-25	-90
-70	87	9	43	57	-25	-90	-80
87	9	43	57	-25	-90	-80	70
9	43	57	-25	-90	-80	70	-87
43	57	-25	-90	-80	70	-87	-9
57	-25	-90	-80	70	-87	-9	-43
-25	-90	-80	70	-87	-9	-43	-57

C_0

180	113	152	-83
113	152	-83	-180
152	-83	-180	-113
-83	-180	-113	-152

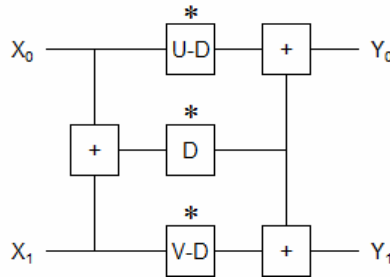
C_1

558	-194
-194	-558

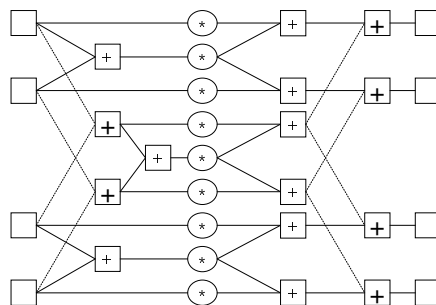
C_2

The C_0 -vector multiplication can be solved adding 3 sub-matrix-vector products starting from C_2 . Each sub-matrix is again anti-circulant and can be solved from 3 lower sub-matrix-vector products, 3^2 for C_1 . The next level is C_0 , the scalar case, where the 3^3 multiplications occur.

For $\begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix} = \begin{bmatrix} U & D \\ D & V \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \end{bmatrix} = \begin{bmatrix} UX_0 + DX_1 \\ DX_0 + VX_1 \end{bmatrix}$ using 4 multiplications (matrix- or scalar) we can construct the basic building block using 3 multiplications:



All the multiplications with this algorithm can be performed parallel without latency and are independent of each other. From 3 scalar building blocks with additional adds the 4-point solver may look like this, which uses 9 multipliers:



For example p_4 has 4 signed base permutation vectors that transforms A_4 to a circulant C_4 . The following table lists $N/2=2$ cyclic permutations of each p_4 :

p_4				K_i									
1	2	-4	3	82	25	46	-68	50	-139	54	-125	232	
2	-4	3	-1	-54	125	-232	-14	-75	93	82	25	46	
3	3	-1	2	54	-125	232	14	75	-93	-82	-25	-46	
4	-1	2	-4	-132	32	-96	32	-50	-39	-196	125	-18	

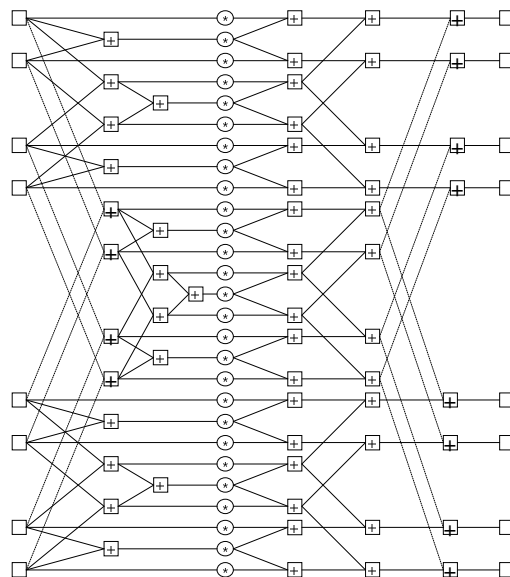
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-2	-4	-3	-1		196	-125	18	-164	75	-57	132	-25	96
1	3	4	2		96	-25	132	-57	75	-164	18	-125	196
3	4	2	-1		-18	125	-196	-39	-50	32	96	-25	132
1	-3	4	-2		46	25	82	93	-75	-14	-232	125	-54
-3	4	-2	-1		232	-125	54	-139	50	-68	46	25	82

The 4-point solver can be *loaded* with any of these constant sets. To improve performance choosing multipliers with least possible terms of dyadic composition can be an advantage. For example:

- $132 = 128 + 4 \rightarrow (x \ll 7) + (x \ll 2)$
- $96 = 64 + 16 \rightarrow (x \ll 6) + (x \ll 4)$
- $68 = 64 + 4 \rightarrow (x \ll 6) + (x \ll 2)$
- $14 = 16 - 2 \rightarrow (x \ll 4) - (x \ll 1)$
- $18 = 16 + 2 \rightarrow (x \ll 4) + (x \ll 1)$
- $7 = 8 - 1 \rightarrow (x \ll 3) - x$
- $112 = 128 - 16 \rightarrow (x \ll 7) - (x \ll 4)$

Similarly, from 3×4 -point solver blocks with additional adds the 8-point solver may look like this, which uses 27 multipliers:



The following table lists the computed 27 multiplier constants according to the first $8/2=4$ sign-cyclic combinations of all 8 signed base permutation p_8 vectors:

1	2	5	-3	-8	7	-4	6		-213	185	-139	250	-123	204	-281	235	-329	79	-18	38	-113				
									43	-133	110	-130	163	55	-149	63	-24	37	62	61	25	3			
2	5	-3	-8	7	-4	6	-1		281	-235	329	-31	112	-125	55	-149	63	-110	130	-163	-3	-87			
									30	-134	167	-101	-61	-25	-3	37	62	65	213	-185	139				
5	-3	-8	7	-4	6	-1	-2		-55	149	-63	24	-37	-62	-61	-25	-3	134	-167	101	-137	80	-71		
									171	-105	166	-213	185	-139	250	-123	204	-281	235	-329					
-3	-8	7	-4	6	-1	-2	-5		61	25	3	-37	-62	-65	-213	185	-139	-171	105	-166	34	-25	95		
									79	-18	38	281	-235	329	-31	112	-125	55	-149	63					
1	-2	5	3	-8	-7	-4	-6		157	-185	231	4	123	-42	189	-235	141	43	18	2	-27	-43	-47	-150	
									130	-97	-243	149	-235	50	-37	136	111	-25	53						
-2	5	3	-8	-7	-4	-6	-1		-189	235	-141	193	-112	99	-243	149	-235	150	-130	97	-177				
									87	-144	200	-167	233	-111	25	-53	161	-62	189	-157	185	-231			

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5	3	-8	-7	-4	-6	-1	2		243 -149 235 -50 37 -136 -111 25 -53 -200 167 -233 23 -80 89 -39 105 -44 157 -185 231 4 123 -42 189 -235 141
3	-8	-7	-4	-6	-1	2	-5		111 -25 53 -161 62 -189 157 -185 231 39 -105 44 -16 25 45 43 18 2 -189 235 -141 193 -112 99 -243 149 -235
1	3	-4	2	8	6	5	-7		283 -75 -151 -239 112 -13 215 11 -123 -116 68 79 82 -25 -65 -92 -55 215 -51 -61 -7 75 -62 143 -31 99 -307
3	-4	2	8	6	5	-7	-1		-215 -11 123 -24 123 -136 -51 -61 -7 92 55 -215 -10 -80 150 167 -7 -72 31 -99 307 44 37 -164 -283 75 151
-4	2	8	6	5	-7	-1	-3		51 61 7 -75 62 -143 31 -99 307 -167 7 72 157 -87 78 -123 44 -92 283 -75 -151 -239 112 -13 215 11 -123
2	8	6	5	-7	-1	-3	4		-31 99 -307 -44 -37 164 283 -75 -151 123 -44 92 34 -43 -14 -116 68 79 -215 -11 123 -24 123 -136 -51 -61 -7
1	-3	-4	-2	8	-6	5	7		133 75 -301 -15 -112 211 237 -11 -101 20 -68 215 32 25 -115 -202 55 105 -173 61 -129 -49 62 19 167 -99 -109
-3	-4	-2	8	-6	5	7	-1		-237 11 101 222 -123 110 -173 61 -129 202 -55 -105 -170 80 -10 153 7 -86 -167 99 109 118 -37 -90 -133 -75 301
-4	-2	8	-6	5	7	-1	3		173 -61 129 49 -62 -19 -167 99 109 -153 -7 86 -17 87 -96 -35 -44 -4 133 75 -301 -15 -112 211 237 -11 -101
-2	8	-6	5	7	-1	3	4		167 -99 -109 -118 37 90 133 75 -301 35 44 4 -52 43 -100 20 -68 215 -237 11 101 222 -123 110 -173 61 -129
1	6	4	7	8	-3	-5	2		-231 185 -157 189 -62 161 -53 25 -111 233 -167 200 -144 87 -177 97 -130 150 -235 149 -243 99 -112 193 -141 235 -189
6	4	7	8	-3	-5	2	-1		53 -25 111 136 -37 50 -235 149 -243 -97 130 -150 -47 -43 -27 2 18 43 141 -235 189 -42 123 4 231 -185 157
4	7	8	-3	-5	2	-1	-6		235 -149 243 -99 112 -193 141 -235 189 -2 -18 -43 -45 -25 16 -44 105 -39 -231 185 -157 189 -62 161 -53 25 -111
7	8	-3	-5	2	-1	-6	-4		-141 235 -189 42 -123 -4 -231 185 -157 44 -105 39 -89 80 -23 233 -167 200 53 -25 111 136 -37 50 -235 149 -243
1	-6	4	-7	8	3	-5	-2		139 -185 213 65 62 37 -3 -25 -61 -101 167 -134 30 -87 -3 -163 130 -110 63 -149 55 -125 112 -31 329 -235 281
-6	4	-7	8	3	-5	-2	-1		3 25 61 62 37 -24 63 -149 55 163 -130 110 -133 43 -113 38 -18 79 -329 235 -281 204 -123 250 -139 185 -213
4	-7	8	3	-5	-2	-1	6		-63 149 -55 125 -112 31 -329 235 -281 -38 18 -79 -95 25 -34 166 -105 171 139 -185 213 65 62 37 -3 -25 -61
-7	8	3	-5	-2	-1	6	-4		329 -235 281 -204 123 -250 139 -185 213 -166 105 -171 71 -80 137 -101 167 -134 3 25 61 62 37 -24 63 -149 55
1	7	-5	-6	-8	-2	4	-3		151 75 -283 -164 37 44 307 -99 31 -72 -7 167 150 -80 -10 -215 55 92 -7 -61 -51 -136 123 -24 123 -11 -215
7	-5	-6	-8	-2	4	-3	-1		-307 99 -31 143 -62 75 -7 -61 -51 215 -55 -92 -65 -25 82 79 68 -116 -123 11 215 -13 112 -239 -151 -75 283
-5	-6	-8	-2	4	-3	-1	-7		7 61 51 136 -123 24 -123 11 215 -79 -68 116 14 43 -34 -92 44 -123 151 75 -283 -164 37 44 307 -99 31
-6	-8	-2	4	-3	-1	-7	5		123 -11 -215 13 -112 239 151 75 -283 92 -44 123 -78 87 -157 -72 -7 167 -307 99 -31 143 -62 75 -7 -61 -51
1	-7	-5	6	-8	2	4	3		301 -75 -133 -90 -37 118 109 99 -167 -86 7 153 -10 80 -170 -105 -55 202 -129 61 -173 110 -123 222 101 11 -237
-7	-5	6	-8	2	4	3	-1		-109 -99 167 19 62 -49 -129 61 -173 105 55 -202 -115 25 32 215 -68 20 -101 -11 237 211 -112 -15 -301 75 133
-5	6	-8	2	4	3	-1	7		129 -61 173 -110 123 -222 -101 -11 237 -215 68 -20 100 -43 52 -4 -44 -35 301 -75 -133 -90 -37 118 109 99 -167
6	-8	2	4	3	-1	7	5		101 11 -237 -211 112 15 301 -75 -133 4 44 35 96 -87 17 -86 7 153 -109 -99 167 19 62 -49 -129 61 -173

The 16-point solver uses 81 multipliers and consists of 3 x 8-point solver blocks with additional adders. Not drawn. The table lists the computed 81 multiplier constants according to the first 16/2=8 sign-cyclic combinations of all 16 signed base permutation p_{16} vectors. Some attention is drawn to certain multiplier sets, where zero, 4, 8, 32 or other dyadic values occur and can be implemented as shift:

1 2 5 14 -9 7 -13 6 -16 15 -12 3 8 -10 4 11	-62 48 402 334 -98 -116 -572 122 -68 -107 95 -297 -148 32 54 131 71 -113 156 -102 342 82 78 -158 -78 -162 252 27 25 -305 -165 42 115 286 -6 16 82 -100 275 39 46 -136 -39 -136 144 -22 32 -350 -99 -68 219 233 85 -123 8 -98 208 -4 14 -114 0 -110 36 -57 105 -253 70 -124 218 -53 201 -175 -112 38 358 116 58 -280 -388 -8 -6
2 5 14 -9 7 -13 6 -16 15 -12 3 8 -10 4 11 -1	572 -122 68 -238 24 -184 156 -102 342 -131 -71 113 -17 103 -59 -49 7 -45 78 162 -252 4 -84 94 -8 98 -208 -286 6 -16 121 36 131 -22 32 -350 39 136 -144 0 -90 8 -60 68 75 -233 -85 123 134 17 96 35 -73 -97 0 110 -36 -4 -96 -78 -112 38 358 53 -201 175 17 77 43 169 -143 -105 388 8 6 -272 50 -286 -62 48 402

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5 14 -9 7 -13 6 -16 15 -12 3 8 -10 4 11 -1 -2	-156 102 -342 -82 -78 158 78 162 -252 49 -7 45 -66 110 -104 53 -91 139 8 -98 208 -4 14 -114 0 -110 36 22 -32 350 99 68 -219 -233 -85 123 60 -68 -75 -60 -22 83 194 -51 21 -35 73 97 169 -56 -1 -286 116 -52 112 -38 -358 -116 -58 280 388 8 6 -169 143 105 186 -66 -62 -441 193 -181 62 -48 -402 -334 98 116 572 -122 68
14 -9 7 -13 6 -16 15 -12 3 8 -10 4 11 -1 -2 -5	-78 -162 252 -4 84 -94 8 -98 208 -53 91 -139 -13 19 35 -57 105 -253 0 110 -36 -4 -96 -78 -112 38 358 233 85 -123 -134 -17 -96 -35 73 97 -194 51 -21 134 -73 104 -25 -5 -22 286 -116 52 -117 60 -53 134 -70 -8 -388 -8 -6 272 -50 286 62 -48 -402 441 -193 181 -255 127 -243 107 -95 297 -572 122 -68 238 -24 184 -156 102 -342
-9 7 -13 6 -16 15 -12 3 8 -10 4 11 -1 -2 -5 -14	-8 98 -208 4 -14 114 0 110 -36 57 -105 253 -70 124 -218 53 -201 175 112 -38 -358 -116 -58 280 388 8 6 35 -73 -97 -169 56 1 286 -116 52 25 5 22 109 -78 82 -92 65 -31 -134 70 8 17 -10 -61 -155 77 -129 -62 48 402 334 -98 -116 -572 122 -68 -107 95 -297 -148 32 54 131 71 -113 156 -102 342 82 78 -158 -78 -162 252
7 -13 6 -16 15 -12 3 8 -10 4 11 -1 -2 -5 -14 9	0 -110 36 4 96 78 112 -38 -358 -53 201 -175 -17 -77 -43 -169 143 105 -388 -8 -6 272 -50 286 62 -48 -402 -286 116 -52 117 -60 53 -134 70 8 92 -65 31 17 -13 51 109 -75 -30 155 -77 129 -138 67 -190 -27 -25 305 572 -122 68 -238 24 -184 156 -102 342 -131 -71 113 -17 103 -59 -49 7 -45 78 162 -252 4 -84 94 -8 98 -208
-13 6 -16 15 -12 3 8 -10 4 11 -1 -2 -5 -14 9 -7	-112 38 358 116 58 -280 -388 -8 -6 169 -143 -105 -186 66 62 441 -193 181 -62 48 402 334 -98 -116 -572 122 -68 134 -70 -8 -17 10 61 155 -77 129 -109 75 30 126 -88 21 -247 142 -160 27 25 -305 -165 42 115 286 -6 16 -156 102 -342 -82 -78 158 78 162 -252 49 -7 45 -66 110 -104 53 -91 139 8 -98 208 -4 14 -114 0 -110 36
6 -16 15 -12 3 8 -10 4 11 -1 -2 -5 -14 9 -7 13	388 8 6 -272 50 -286 -62 48 402 -441 193 -181 255 -127 243 -107 95 -297 572 -122 68 -238 24 -184 156 -102 342 -155 77 -129 138 -67 190 27 25 -305 247 -142 160 -121 54 -139 82 -100 275 -286 6 -16 121 36 131 -22 32 -350 -78 -162 252 -4 84 -94 8 -98 208 -53 91 -139 -13 19 35 -57 105 -253 0 110 -36 -4 -96 -78 -112 38 358
1 -2 5 -14 -9 -7 -13 -6 -16 -15 -12 -3 8 10 4 -11	34 -48 498 138 98 -312 -328 -122 176 83 -95 -107 -84 -32 118 273 -71 29 -48 102 138 238 -78 -2 -402 162 -72 77 -25 -255 -81 -42 199 274 6 4 -118 100 75 131 -46 -44 -311 136 -128 42 -32 -286 -235 68 83 403 -85 47 -188 98 12 24 -14 -86 -220 110 -184 153 -105 -43 -178 124 -30 349 -201 227 -36 -38 434 232 -58 -164 -404 8 -22
-2 5 -14 -9 -7 -13 -6 -16 -15 -12 -3 8 10 4 -11 -1	328 122 -176 -190 -24 -136 -48 102 138 -273 71 -29 189 -103 147 -35 -7 -31 402 -162 72 -164 84 -74 188 -98 -12 -274 -6 -4 193 -36 203 42 -32 -286 311 -136 128 -180 90 -172 76 -68 211 -403 85 -47 168 -17 130 -111 73 -243 220 -110 184 -196 96 -270 -36 -38 434 -349 201 -227 171 -77 197 -117 143 -391 404 -8 22 -172 -50 -186 34
5 -14 -9 -7 -13 -6 -16 -15 -12 -3 8 10 4 -11 -1 2	48 -102 -138 -238 78 2 402 -162 72 35 7 31 154 -110 116 -129 91 -43 -188 98 12 24 -14 -86 -220 110 -184 -42 32 286 235 -68 -83 -403 85 -47 -76 68 -211 -104 22 39 92 51 -81 111 -73 243 57 56 -113 -54 -116 180 36 38 -434 -232 58 164 404 -8 22 117 -143 391 54 66 -194 -55 -193 205 -34 48 -498 -138 -98 312 328 122 -176
-14 -9 -7 -13 -6 -16 -1 5 -12 -3 8 10 4 -11 -1 2 -5	-402 162 -72 164 -84 74 -188 98 12 129 -91 43 25 -19 73 153 -105 -43 220 -110 184 -196 96 -270 -36 -38 434 403 -85 47 -168 17 -130 111 -73 243 -92 -51 81 -12 73 -42 -35 5 -32 54 116 -180 3 -60 67 -6 70 -148 -404 8 -22 172 50 186 -34 48 -498 55 193 -205 -1 -127 11 -83 95 107 -328 -122 176 190 24 136 48 -102 -138
-9 -7 -13 -6 -16 -15 -1 2 -3 8 10 4 -11 -1 2 -5 14	188 -98 -12 -24 14 86 220 -110 184 -153 105 43 178 -124 30 -349 201 -227 36 38 -434 -232 58 164 404 -8 22 -111 73 -243 -57 -56 113 54 116 -180 35 -5 32 -47 78 -74 38 -65 99 6 -70 148 -3 10 -81 -1 -77 25 34 -48 498 138 98 -312 -328 -122 176 83 -95 -107 -84 -32 118 273 -71 29 -48 102 138 238 -78 -2 -402 162 -72
-7 -13 -6 -16 -15 -12 - 3 8 10 4 -11 -1 2 -5 14 9	-220 110 -184 196 -96 270 36 38 -434 349 -201 227 -171 77 -197 117 -143 391 -404 8 -22 172 50 186 -34 48 -498 -54 -116 180 -3 60 -67 6 -70 148 -38 65 -99 -9 13 25 -41 75 -180 1 77 -25 -4 -67 -56 -77 25 255 328 122 -176 -190 -24 -136 -48 102 138 -273 71 -29 189 -103 147 -35 -7 -31 402 -162 72 -164 84 -74 188 -98 -12
-13 -6 -16 -15 -12 -3 8 10 4 -11 -1 2 -5 14 9 7	-36 -38 434 232 -58 -164 -404 8 -22 -117 143 -391 -54 -66 194 55 193 -205 34 -48 498 138 98 -312 -328 -122 176 -6 70 -148 3 -10 81 1 77 -25 41 -75 180 -50 88 -155 37 -142 124 77 -25 -255 -81 -42 199 274 6 4 48 -102 -138 -238 78 2 402 -162 72 35 7 31 154 -110 116 -129 91 -43 -188 98 12 24 -14 -86 -220 110 -184
-6 -16 -15 -12 -3 8 10 4 -11 -1 2 -5 14 9 7 13	404 -8 22 -172 -50 -186 34 -48 498 -55 -193 205 1 127 -11 83 -95 -107 328 122 -176 -190 -24 -136 -48 102 138 -1 -77 25 4 67 56 77 -25 -255 -37 142 -124 -13 -54 -31 -118 100 75 -274 -6 -4 193 -36 203 42 -32 -286 -402 162 -72 164 -84 74 -188 98 12 129 -91 43 25 -19 73 153 -105 -43 220 -110 184 -196 96 -270 -36 -38 434
1 3 13 2 8 -6 5 -10 16 14 4 -15 -9 11 12 7	96 -40 -92 -328 230 -196 596 -464 358 81 3 -53 171 -127 221 -187 237 -215 -136 30 130 -132 204 -292 186 -346 438 -44 62 -44 133 -86 115 -238 220 -180 -72 -27 155 -35 73 -163 33 -161 220 11 29 -121 94 -163 186 -237 329 -403 -8 -84 180 62 -58 -34 -120 24 2 63 51 -257 -101 -19 105 121 85 -225 114 -88 112 -56 122 -80 288 -312
3 13 2 8 -6 5 -10 16 14 4 -15 -9 11 12 7 -1	-596 464 -358 268 -234 162 -136 30 130 187 -237 215 -16 110 6 55 -33 -77 -186 346 -438 54 -142 146 8 84 -180 238 -220 180 -105 134 -65 11 29 -121 -33 161 -220 -2 -88 57 61 -2 -34 237 -329 403 -143 166 -217 -52 -22 136 120 -24 -2 -58 -34 -32 114 -88 112 -121 -85 225 20 66 -120 -177 37 145 -288 312 -368 232 -190 288 96 -40 -
13 2 8 -6 5 -10 16 14 4 -15 -9 11 12 7 -1 -3	136 -30 -130 132 -204 292 -186 346 -438 -55 33 77 39 77 -71 -1 -109 223 -8 -84 180 62 -58 -34 -120 24 2 -11 -29 121 -94 163 -186 237 -329 403 -61 2 34 59 -90 23 -204 168 -183 52 22 -136 -195 144 -81 358 -244 178 -114 88 -112 56 -122 80 -288 312 -368 177 -37 -145 -157 103 25 409 -227 143 -96 40 92 328 -230 196 -596 464 -358

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2 8 -6 5 -10 16 14 4 -15 -9 11 12 7 -1 -3 -13	186 -346 438 -54 142 -146 -8 -84 180 1 109 -223 38 -32 152 63 51 -257 120 -24 -2 -58 -34 -32 114 -88 112 -237 329 -403 143 -166 217 52 22 -136 204 -168 183 -145 78 -160 9 -24 102 -358 244 -178 163 -100 97 -125 59 9 288 -312 368 -232 190 -288 -96 40 92 -409 227 -143 252 -124 168 -81 -3 53 596 -464 358 -268 234 -162 136 -30
8 -6 5 -10 16 14 4 -15 -9 11 12 7 -1 -3 -13 -2	8 84 -180 -62 58 34 120 -24 -2 -63 -51 257 101 19 -105 -121 -85 225 -114 88 -112 56 -122 80 -288 312 -368 -52 -22 136 195 -144 81 -358 244 -178 -9 24 -102 -136 54 -58 154 -76 -5 125 -59 -9 38 -41 106 51 17 -35 96 -40 -92 -328 230 -196 596 -464 358 81 3 -53 171 -127 221 -187 237 -215 -136 30 130 -132 204 -292 186 -346 438
-6 5 -10 16 14 4 -15 -9 11 12 7 -1 -3 -13 -2 -8	-120 24 2 58 34 32 -114 88 -112 121 85 -225 -20 -66 120 177 -37 -145 288 -312 368 -232 190 -288 -96 40 92 358 -244 178 -163 100 -97 125 -59 -9 -154 76 5 18 -22 -63 -116 35 111 -51 -17 35 89 -24 71 44 -62 44 -596 464 -358 268 -234 162 -136 30 130 187 -237 215 -16 110 6 55 -33 -77 -186 346 -438 54 -142 146 8 84 -180
5 -10 16 14 4 -15 -9 11 12 7 -1 -3 -13 -2 -8 6	114 -88 112 -56 122 -80 288 -312 368 -177 37 145 157 -103 -25 -409 227 -143 96 -40 -92 -328 230 -196 596 -464 358 -125 59 9 -38 41 -106 -51 -17 35 116 -35 -111 -98 13 48 205 -59 -40 -44 62 -44 133 -86 115 -238 220 -180 136 -30 -130 132 -204 292 -186 346 -438 -55 33 77 39 77 -71 -1 -109 223 -8 -84 180 62 -58 -34 -120 24 2
-10 16 14 4 -15 -9 11 12 7 -1 -3 -13 -2 -8 6 -5	-288 312 -368 232 -190 288 96 -40 -92 409 -227 143 -252 124 -168 81 3 -53 -596 464 -358 268 -234 162 -136 30 130 51 17 -35 -89 24 -71 -44 62 -44 -205 59 40 107 -46 8 -72 -27 155 238 -220 180 -105 134 -65 11 29 -121 186 -346 438 -54 142 -146 -8 -84 180 1 109 -223 38 -32 152 63 51 -257 120 -24 -2 -58 -34 -32 114 -88 112
1 -3 13 -2 8 6 5 10 16 -14 4 15 -9 -11 12 -7	16 40 -172 132 -230 264 -332 464 -570 87 -3 -47 -83 127 -33 287 -237 259 -76 -30 190 276 -204 116 -506 346 -254 80 -62 80 -39 86 -57 202 -220 260 -126 27 101 111 -73 -17 -289 161 -102 69 -29 -63 -232 163 -140 421 -329 255 -176 84 12 -54 58 -150 -72 -24 50 165 -51 -155 -139 19 67 291 -85 -55 -62 88 -64 188 -122 164 -336 312
-3 13 -2 8 6 5 10 16 -14 4 15 -9 -11 12 -7 -1	332 -464 570 -200 234 -306 -76 -30 190 -287 237 -259 204 -110 226 -11 33 -143 506 -346 254 -230 142 -138 176 -84 -12 -202 220 -260 163 -134 203 69 -29 -63 289 -161 102 -178 88 -119 57 2 -38 -421 329 -255 189 -166 115 -96 22 92 72 24 -50 -126 34 -100 -62 88 -64 -291 85 55 152 -66 12 -103 -37 219 336 -312 256 -148 190 -92 16
13 -2 8 6 5 10 16 -14 4 15 -9 -11 12 -7 -1 3	76 30 -190 -276 204 -116 506 -346 254 11 -33 143 193 -77 83 -219 109 5 -176 84 12 -54 58 -150 -72 -24 50 -69 29 63 232 -163 140 -421 329 -255 -57 -2 38 -121 90 -157 132 -168 153 96 -22 -92 93 -144 207 -130 244 -310 62 -88 64 -188 122 -164 336 -312 256 103 37 -219 49 -103 231 -45 227 -311 -16 -40 172 -132 230 -264 332 -464
-2 8 6 5 10 16 -14 4 15 -9 -11 12 -7 -1 3 -13	-506 346 -254 230 -142 138 -176 84 12 219 -109 -5 -26 32 88 165 -51 -155 72 24 -50 -126 34 -100 -62 88 -64 421 -329 255 -189 166 -115 96 -22 -92 -132 168 -153 11 -78 -4 -39 24 54 130 -244 310 -37 100 -103 -7 -59 127 -336 312 -256 148 -190 92 -16 -40 172 45 -227 311 4 124 -80 -87 3 47 -332 464 -570 200 -234 306 76 30 -190
8 6 5 10 16 -14 4 15 -9 -11 12 -7 -1 3 -13 2	176 -84 -12 54 -58 150 72 24 -50 -165 51 155 139 -19 -67 -291 85 55 62 -88 64 -188 122 -164 336 -312 256 -96 22 92 -93 144 -207 130 -244 310 39 -24 -54 -28 -54 50 2 76 -157 7 59 -127 -44 41 24 85 -17 -1 16 40 -172 132 -230 264 -332 464 -570 87 -3 -47 -83 127 -33 287 -237 259 -76 -30 190 276 -204 116 -506 346 -254
6 5 10 16 -14 4 15 -9 -11 12 -7 -1 3 -13 2 -8	-72 -24 50 126 -34 100 62 -88 64 291 -85 -55 -152 66 -12 103 37 -219 -336 312 -256 148 -190 92 -16 -40 172 -130 244 -310 37 -100 103 7 59 -127 -2 -76 157 -26 22 -107 -46 -35 181 -85 17 1 41 24 23 -80 62 -80 332 -464 570 -200 234 -306 -76 -30 190 -287 237 -259 204 -110 226 -11 33 -143 506 -346 254 -230 142 -138 176 -84 -12
5 10 16 -14 4 15 -9 -11 12 -7 -1 3 -13 2 -8 -6	-62 88 -64 188 -122 164 -336 312 -256 -103 -37 219 -49 103 -231 45 -227 311 16 40 -172 132 -230 264 -332 464 -570 -7 -59 127 44 -41 -24 -85 17 1 46 35 -181 -72 -13 74 87 59 -158 80 -62 80 -39 86 -57 202 -220 260 76 30 -190 -276 204 -116 506 -346 254 11 -33 143 193 -77 83 -219 109 5 -176 84 12 -54 58 -150 -72 -24 50
10 16 -14 4 15 -9 -11 12 -7 -1 3 -13 2 -8 -6 -5	336 -312 256 -148 190 -92 16 40 -172 -45 227 -311 -4 -124 80 87 -3 -47 332 -464 570 -200 234 -306 -76 -30 190 85 -17 -1 -41 -24 -23 80 -62 80 -87 -59 158 15 46 -84 -126 27 101 -202 220 -260 163 -134 203 69 -29 -63 -506 346 -254 230 -142 138 -176 84 12 219 -109 -5 -26 32 88 165 -51 -155 72 24 -50 -126 34 -100 -62 88 -64
1 6 4 -7 -8 14 -12 -2 16 -11 -13 -10 9 -3 5 15	48 -70 528 152 84 -306 -452 -6 54 -221 213 -427 -17 -103 197 79 135 -175 102 -54 328 198 -24 -76 -200 -74 164 -53 109 -373 -46 -67 218 207 57 -41 80 -101 273 69 13 -103 -81 -91 96 66 -50 -274 -227 60 97 328 -4 -30 58 -148 218 -60 50 -130 38 -108 28 61 -11 -119 -121 77 9 83 47 -17 -234 154 220 256 -96 -118 -456 82 -104
6 4 -7 -8 14 -12 -2 16 -11 -13 -10 9 -3 5 15 -1	452 6 -54 -300 78 -252 102 -54 328 -79 -135 175 62 32 22 119 -159 99 200 74 -164 -2 -98 88 -58 148 -218 -207 -57 41 161 -10 177 66 -50 -274 81 91 -96 -12 -78 -7 -146 151 1 -328 4 30 101 56 67 5 -39 -155 -38 108 -28 -22 -58 -102 -234 154 220 -83 -47 17 -38 124 -8 173 -143 -101 456 -82 104 -200 -14 -222 48 -70 528
4 -7 -8 14 -12 -2 16 -11 -13 -10 9 -3 5 15 -1 -6	-102 54 -328 -198 24 76 200 74 -164 -119 159 -99 181 -127 121 -121 61 -11 58 -148 218 -60 50 -130 38 -108 28 -66 50 274 227 -60 -97 -328 4 30 146 -151 -1 -158 73 -6 247 -95 66 -5 39 155 106 17 -88 -245 51 13 234 -154 -220 -256 96 118 456 -82 104 -173 143 101 135 -19 -109 -373 129 -121 -48 70 -528 -152 -84 306 452 6 -54
-7 -8 14 -12 -2 16 -11 -13 -10 9 -3 5 15 -1 -6 -4	-200 -74 164 2 98 -88 58 -148 218 121 -61 11 60 -66 110 61 -11 -119 -38 108 -28 -22 -58 -102 -234 154 220 328 -4 -30 -101 -56 -67 -5 39 155 -247 95 -66 89 -22 60 -141 112 -154 245 -51 -13 -139 68 -75 168 -104 54 -456 82 -104 200 14 222 -48 70 -528 373 -129 121 -238 110 -230 221 -213 427 -452 -6 54 300 -78 252 -102 54 -328

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-8 14 -12 -2 16 -11 -13 -10 9 -3 5 15 -1 -6 -4 7	-58 148 -218 60 -50 130 -38 108 -28 -61 11 119 121 -77 -9 -83 -47 17 234 -154 -220 -256 96 118 456 -82 104 5 -39 -155 -106 -17 88 245 -51 -13 141 -112 154 -52 90 -94 2 -44 79 -168 104 -54 29 -36 -21 -128 78 -134 48 -70 528 152 84 -306 -452 -6 54 -221 213 -427 -17 -103 197 79 135 -175 102 -54 328 198 -24 -76 -200 -74 164
14 -12 -2 16 -11 -13 -10 9 -3 5 15 -1 -6 -4 7 8	38 -108 28 22 58 102 234 -154 -220 83 47 -17 38 -124 8 -173 143 101 -456 82 -104 200 14 222 -48 70 -528 -245 51 13 139 -68 75 -168 104 -54 -2 44 -79 -50 46 -15 27 8 -100 128 -78 134 -99 42 -155 53 -109 373 452 6 -54 -300 78 -252 102 -54 328 -79 -135 175 62 32 22 119 -159 99 200 74 -164 -2 -98 88 -58 148 -218
-12 -2 16 -11 -13 -10 9 -3 5 15 -1 -6 -4 7 8 -14	-234 154 220 256 -96 -118 -456 82 -104 173 -143 -101 -135 19 109 373 -129 121 48 -70 528 152 84 -306 -452 -6 54 168 -104 54 -29 36 21 128 -78 134 -27 -8 100 -23 54 -115 -126 34 -55 -53 109 -373 -46 -67 218 207 57 -41 -102 54 -328 -198 24 76 200 74 -164 -119 159 -99 181 -127 121 -121 61 -11 58 -148 218 -60 50 -130 38 -108 2
-2 16 -11 -13 -10 9 -3 5 15 -1 -6 -4 7 8 -14 12	456 -82 104 -200 -14 -222 48 -70 528 -373 129 -121 238 -110 230 -221 213 -427 452 6 -54 -300 78 -252 102 -54 328 -128 78 -134 99 -42 155 -53 109 -373 126 -34 55 -149 88 -170 80 -101 273 -207 -57 41 161 -10 177 66 -50 -274 -200 -74 164 2 98 -88 58 -148 218 121 -61 11 60 -66 110 61 -11 -119 -38 108 -28 -22 -58 -102 -234 154
1 -6 4 7 -8 -14 -12 2 16 11 -13 10 9 3 5 -15	-92 70 388 320 -84 -138 -464 6 42 205 -213 -1 -223 103 -9 349 -135 95 -6 54 220 150 24 -124 -348 74 16 165 -109 -155 -180 67 84 321 -57 73 -122 101 71 95 -13 -77 -263 91 -86 -34 50 -374 -107 -60 217 320 4 -38 -238 148 -78 40 -50 -30 -178 108 -188 39 11 -141 33 -77 163 177 -47 77 74 -154 528 64 96 -310 -292 -82 60
-6 4 7 -8 -14 -12 2 16 11 -13 10 9 3 5 -15 -1	464 -6 -42 -144 -78 -96 -6 54 220 -349 135 -95 126 -32 86 -199 159 -219 348 -74 -16 -198 98 -108 238 -148 78 -321 57 -73 141 10 157 -34 50 -374 263 -91 86 -168 78 -163 156 -151 303 -320 -4 38 213 -56 179 -73 39 -233 178 -108 188 -138 58 -218 74 -154 528 -177 47 -77 210 -124 240 -113 143 -387 292 82 -60 -228 14 -250 -92 70
4 7 -8 -14 -12 2 16 11 -13 10 9 3 5 -15 -1 6	6 -54 -220 -150 -24 124 348 -74 -16 199 -159 219 -73 127 -133 1 -61 111 -238 148 -78 40 -50 -30 -178 108 -188 34 -50 374 107 60 -217 -320 -4 38 -156 151 -303 -12 -73 140 57 95 -124 73 -39 233 140 -17 -54 -143 -51 115 -74 154 -528 -64 -96 310 292 82 -60 113 -143 387 97 19 -147 -115 -129 137 92 -70 -388 -320 84 138 464 -6 -4
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