Exact and Efficient Inference for Partial Bayes Problems

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Introduction

- Bayesian model is the "golden rule" for analyzing data that have prior information.
- However, the assumption of a fully known prior distribution is sometimes too strong.
- Many useful models can be viewed as **Partial Bayes Problems**, in which the prior distribution is *partially* specified.
- We systematically study a class of Partial Bayes problems, and develop interval estimators for the parameter of interest.
- Theoretical analysis and simulation results demonstrate the exactness and efficiency of the proposed inference results.

Example: Bayesian Hierarchical Model

- One typical Partial Bayes problem is the Bayesian hierarchical model with unknown hyper-parameters in the prior.

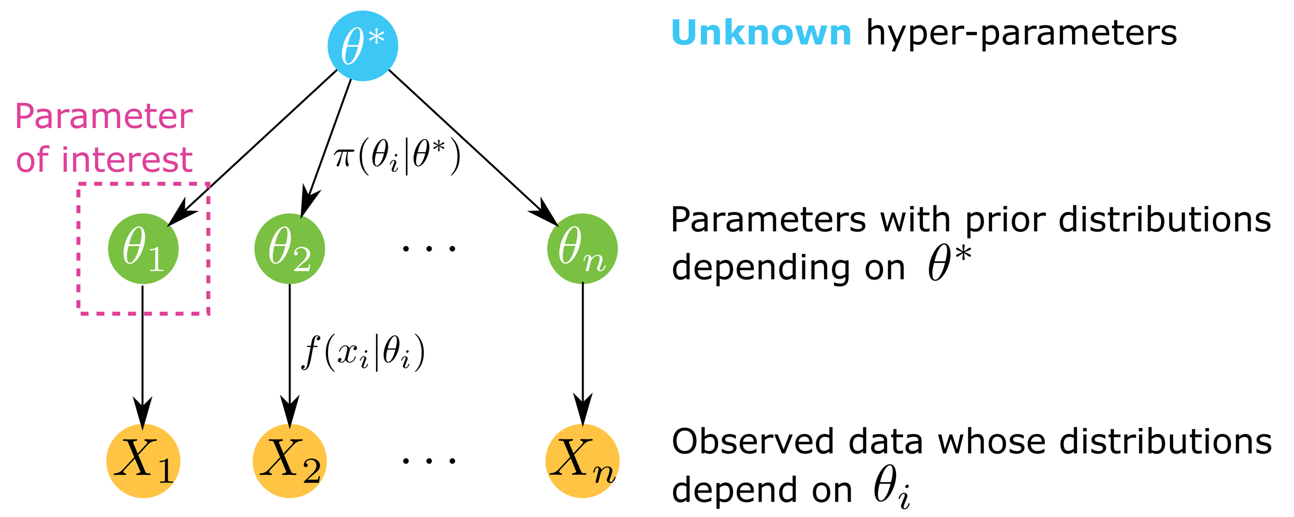

Figure 1: A typical Partial Bayes problem based on Bayesian hierarchical model

General Model Setting and Inference Objective

Component	Formula
Sampling model	$X\|\theta \sim f(x\|\theta)$
Parameter partition	$\theta = (\tilde{\theta}, \theta^*)$
Partial prior	$\tilde{\theta}\|\theta^* \sim \pi(\tilde{\theta}\|\theta^*)$
Component without prior	θ^*
Parameter of interest	$\eta = h(\tilde{\theta})$

- Objective of inference: To construct an interval estimator $C_\alpha(X)$ for η that satisfies the validity condition given below.
- Formally, we use the definition from Morris (1983):

$$P(\eta \in C_\alpha(X)) \geq 1 - \alpha, \text{ for all } \theta^*,$$

where P is computed over the joint distribution of (θ, X).

Existing Methods and Challenges

- Full Bayesian approach (Deely and Lindley, 1981).
- Empirical Bayes (Robbins, 1956; Efron and Morris, 1971, 1972, 1973, 1975).
- Empirical Bayes confidence interval (Morris, 1983; Carlin and Gelfand, 1990).
- Confidence distribution (Xie et al, 2011; Xie et al, 2013).
- However, preserving validity is highly **non-trivial**.

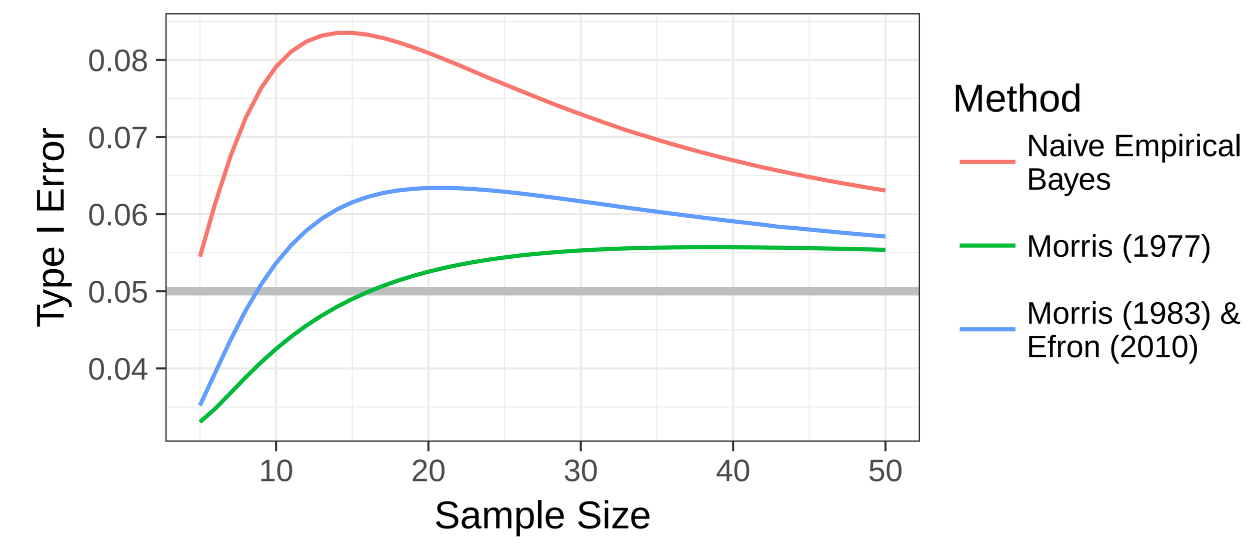

Figure 2: Type I error of existing methods for a Bayesian hierarchical model example in Efron (2010).

The Proposed Solution

- We systematically analyze a class of Partial Bayes problems using **Inferential Models** (Martin and Liu, 2013, 2015a, b), a novel framework for exact statistical inference.
- For illustration purpose, we use a minimal example to demonstrate the basic idea.

Two-Observation Normal-Normal Model

- $X_i|\mu_i \sim N(\mu_i, 1), i = 1, 2$. X_1, X_2 are independent. μ_1, μ_2 are unknown means.
- $\mu_i \sim N(\mu, 1)$ is the independent common prior, where μ is **unknown**.
- Want to make inference about μ_1 (or μ_2).

- Inferential Models follow a three-step procedure:

Association Step

Connect data, parameters, and unobserved auxiliary random variables through a series of association functions.

$$\begin{cases} X_1 = \mu_1 + e_1 = (\mu + \varepsilon_1) + e_1 \\ X_2 = \mu_2 + e_2 = (\mu + \varepsilon_2) + e_2 \end{cases} \Leftrightarrow \begin{cases} X_1 &= \mu_1 + e_1 \\ X_2 - X_1 &= \varepsilon_2 - \varepsilon_1 + e_2 - e_1 \end{cases}, \quad e_i, \varepsilon_i \overset{iid}{\sim} N(0,1)$$

- $\mu_1 = \Theta_x(e_1) = x - e_1$: If the true value of e_1 is known, so is μ_1.

Prediction Step

Predict the unobserved auxiliary random variables with a nested random set.

- $e_1|\{\varepsilon_2 - \varepsilon_1 + e_2 - e_1 = x_2 - x_1\} \sim N\left(\frac{1}{4}(x_1 - x_2), \frac{3}{4}\right)$.
- Construct the random set as

$$\mathcal{S} = \left(\frac{1}{4}(x_1 - x_2) - \frac{\sqrt{3}}{2}|Z|, \ \frac{1}{4}(x_1 - x_2) + \frac{\sqrt{3}}{2}|Z|\right), \quad Z \sim N(0, 1).$$

- $\mathcal{S}$ can cover the true value of e_1 with a high probability.

Combination Step

Transform the uncertainty from the space of auxiliary variable to the space of parameter.

- $\mathcal{S}$ Deduces a predictive random set for the parameter: $\Theta_x(\mathcal{S}) = \bigcup_{e_1 \in \mathcal{S}} \Theta_x(e_1)$.
- Compute the plausibility function: $\mathsf{pl}_x(\theta) = 1 - P\{\Theta_x(\mathcal{S}) \subseteq \{\theta\}^c | \Theta_x(\mathcal{S}) \neq \varnothing\}$.
- The interval estimator is obtained as $C_\alpha(x) = \{\theta : \mathsf{pl}_x(\theta) > \alpha\}$, *i.e.*,

$$C_\alpha(X) = \left(\frac{3}{4}X_1 + \frac{1}{4}X_2\right) \pm \frac{\sqrt{3}}{2} z_{\alpha/2}$$

- In contrast, naive empirical Bayes provides the interval estimator

$$C_\alpha(X) = \left(\frac{3}{4}X_1 + \frac{1}{4}X_2\right) \pm \frac{\sqrt{2}}{2} z_{\alpha/2}$$

General Results

- We have derived solutions to a general class of Partial Bayes models with theoretical guarantees. In plain words,
- Exactness: Partial Bayes solution is valid.
- Optimality: If the prior is fully given, Partial Bayes matches the Bayesian solution.
- Efficiency: If the prior is partially given, then under mild conditions, Partial Bayes solution is a good approximation to the Bayesian solution.

Simulation Study

- **Normal hierarchical model**: $X_i|\mu_i \overset{indep}{\sim} N(\mu_i, \sigma^2)$ with a common prior $\mu_i \overset{iid}{\sim} N(\mu, \tau^2)$, $i = 1, \ldots, n$. Both μ and τ^2 are unknown. μ_1 is the parameter of interest.

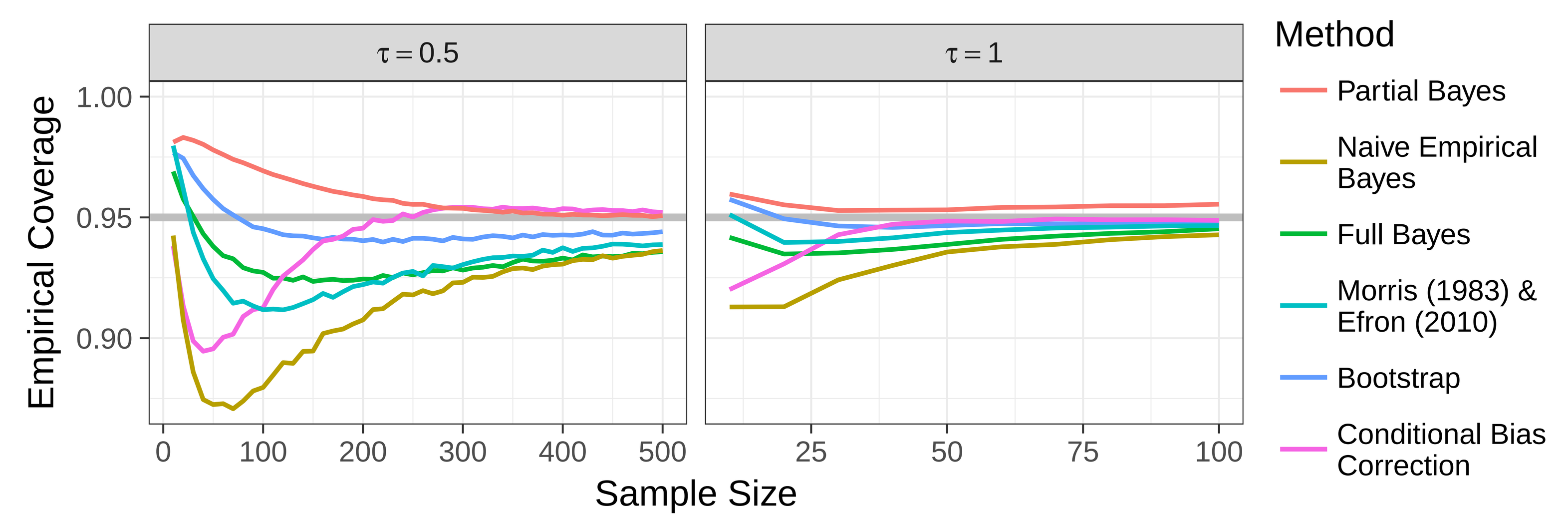

Figure 3: The empirical coverage percentage given by different methods for the normal hierarchical model. Only the Partial Bayes solution guarantees the nominal coverage rate for all sample sizes.

- **Poisson hierarchical model**: $X_i|\lambda_i \overset{indep}{\sim} Pois(\lambda_i)$ with a common prior $\lambda_i \overset{iid}{\sim} \theta Gamma(s)$, $i = 1, \ldots, n$. Assume that s is known but θ is unknown. λ_1 is the parameter of interest.

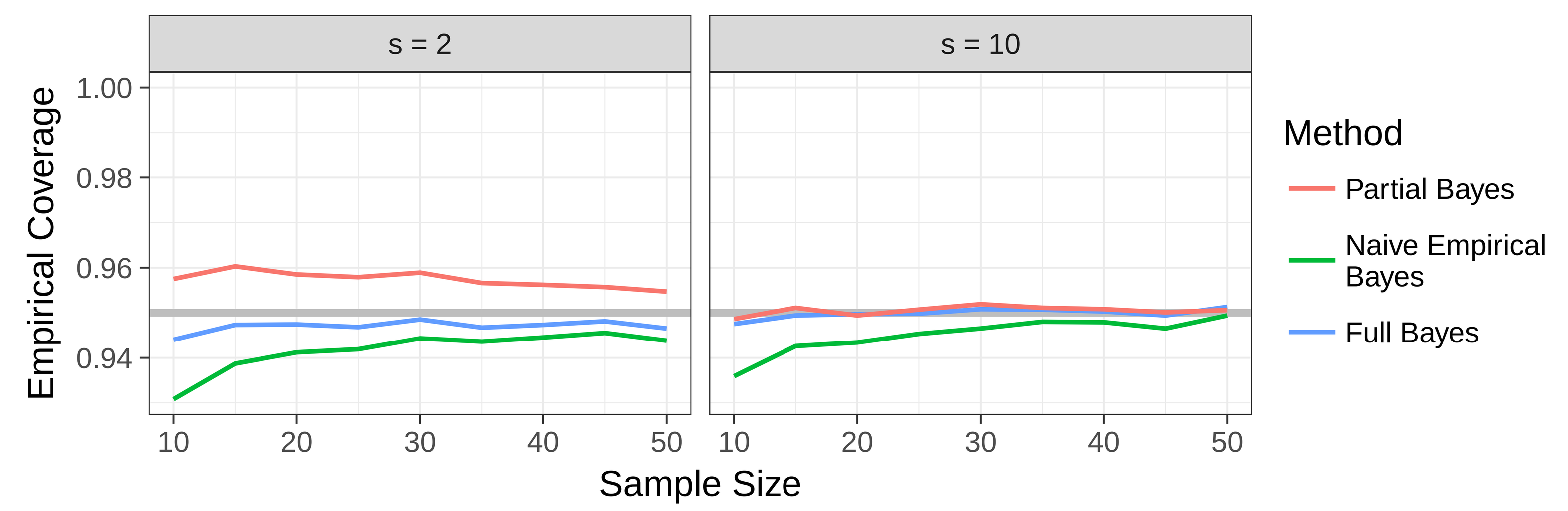

Figure 4: The empirical coverage percentage given by different methods for the Poisson hierarchical model, showing that the Partial Bayes solution preserves the nominal coverage rate.

Conclusion

- Partial Bayes models allow for more flexibility in Bayesian data analysis.
- Solving Partial Bayes problems with validity guarantee is a challenging task, and is usually non-trivial.
- We have obtained general solutions to Partial Bayes problems using the Inferential Model framework.
- Theoretical analysis and simulation results demonstrate the superiority of Partial Bayes solutions.

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