Finer separations between shallow arithmetic circuits

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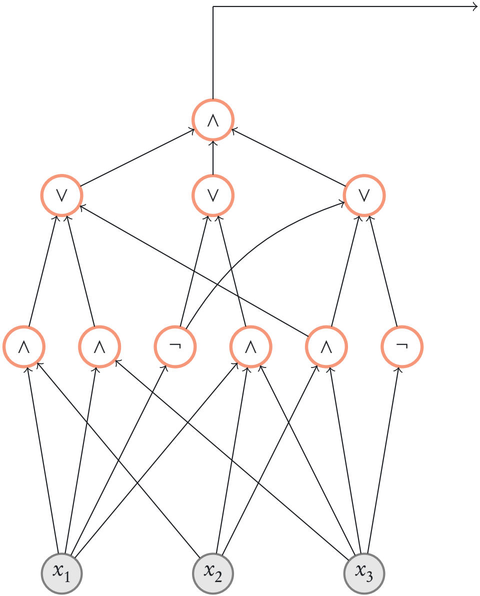
(Work done while in Tel Aviv University)

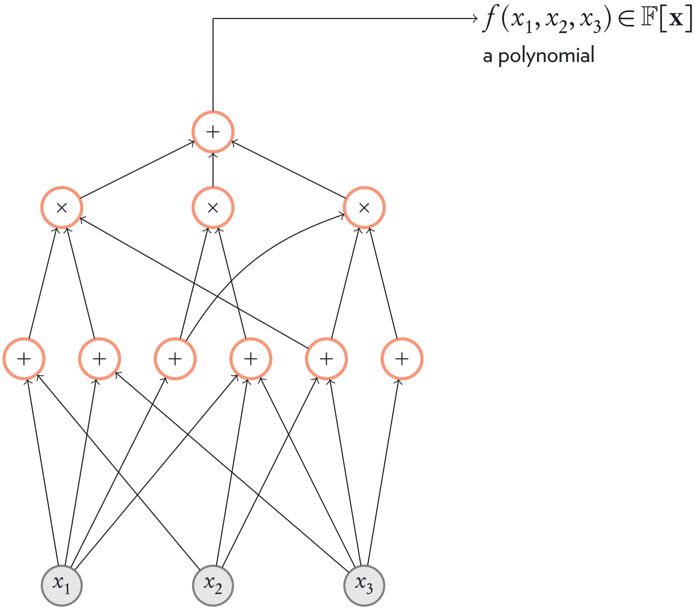
Boolean circuits



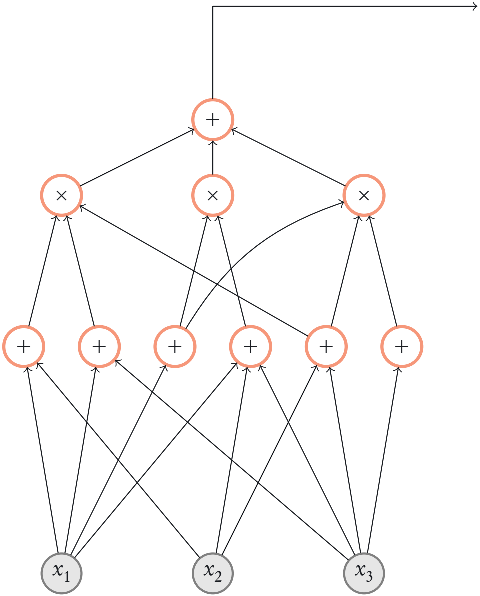
$f(x_1, x_2, x_3) : \{0,1\}^n \to \{0,1\}$

a boolean function

Arithmetic circuits



Arithmetic circuits



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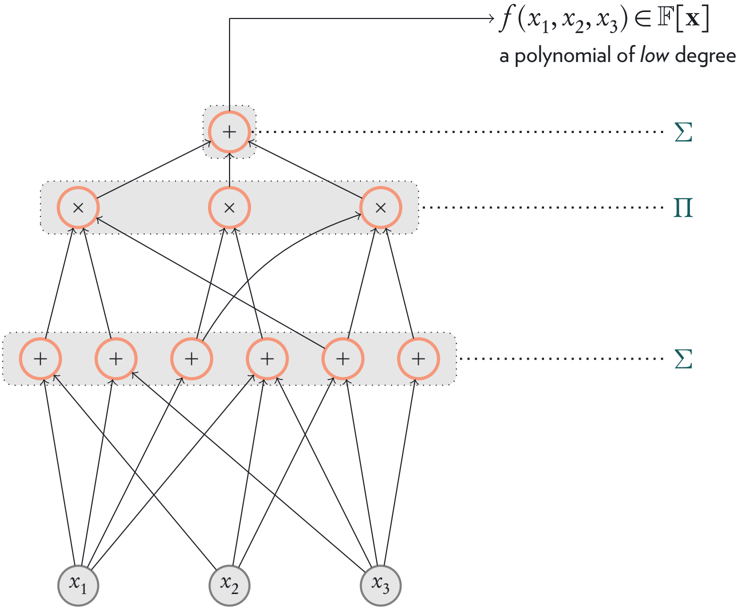
a polynomial of *low* degree

Arithmetic circuits



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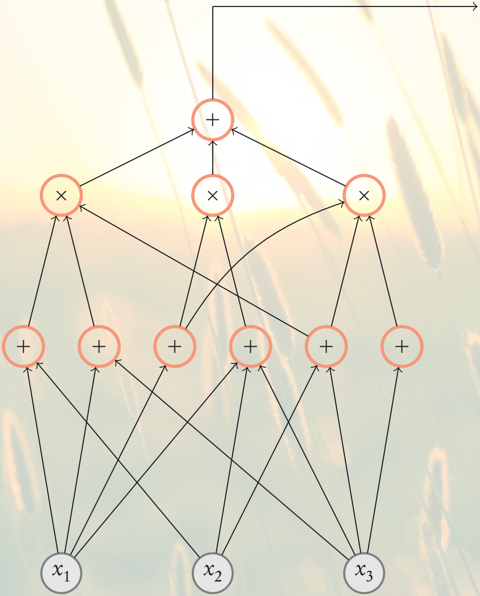

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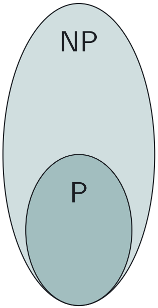
The Open Problem(s)



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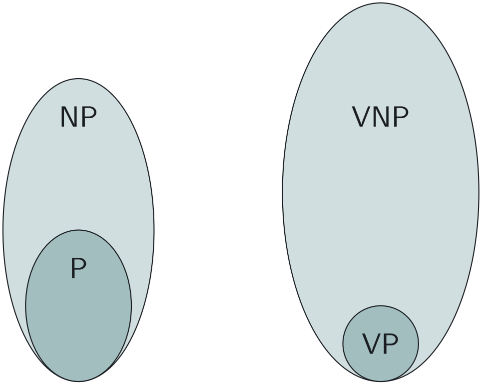

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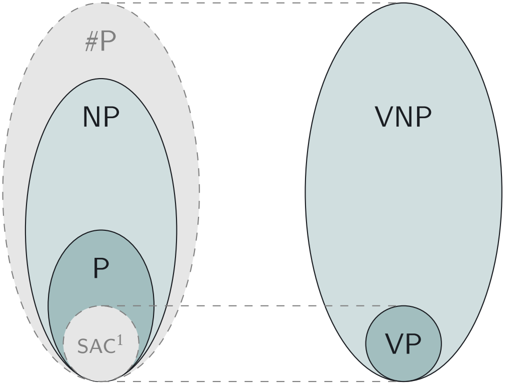

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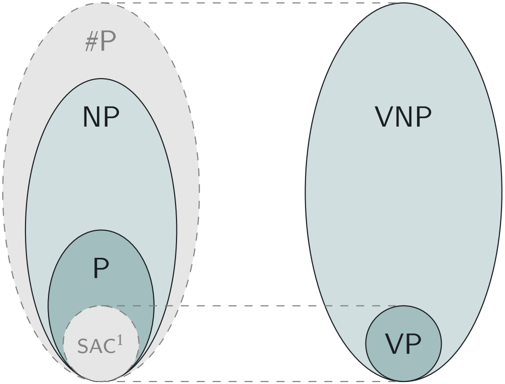

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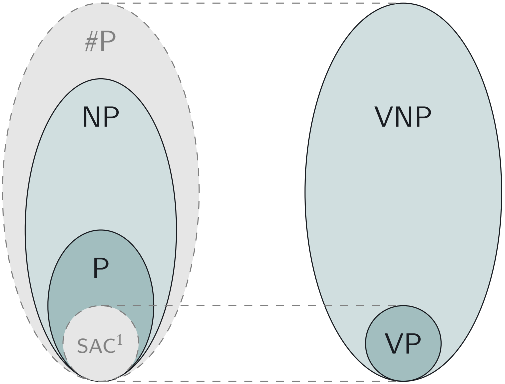


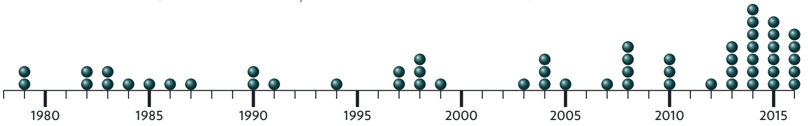
VP $\neq$ VNP is simpler to prove than P $\neq$ NP.

Ultimate goal: Find an explicit n-variate degree d polynomial that requires large arithmetic circuits to compute it.

Recent activity in algebraic complexity

A recent surge in optimism in the field.

Someone even conjectured that VP $\neq$ VNP would be resolved by 2018...



Each ● is one result in algebraic complexity lower bounds.

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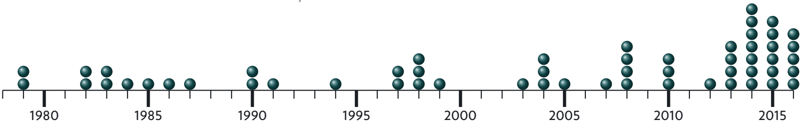

Each ● is one result in algebraic complexity lower bounds.

Point of attack: Lower bounds for very shallow circuits

Completely different situation from the boolean world!

Depth Reduction

Theorem ([Agrawal-Vinay + Koiran, Tavenas])

Can be computed by		*Can be computed by*
arithmetic circuits	$\longrightarrow$	*depth-4 circuits**
of "small" size		*of "not-too-large" size*

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Can be computed by *arithmetic circuits* *of* $\mathrm{poly}(n,d)$ *size*	$\longrightarrow$	*Can be computed by* *depth-4 circuits** *of* $n^{O(\sqrt{d})}$ *size*

(Or)

Cannot be computed by *arithmetic circuits* *of* $\mathrm{poly}(n,d)$ *size*	$\longleftarrow$	*Cannot be computed by* *depth-4 circuits** *of* $n^{O(\sqrt{d})}$ *size*

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Goal: To prove an $n^{\omega(\sqrt{d})}$ lower bound for depth-4 circuits*.

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... and the deluge began –
[Fournier-Limaye-Malod-Srinivasan], [Kayal-Saha-Saptharishi], [Chillara-Mukhopadhyay], [Kumar-Saraf], [Kayal-Limaye-Saha-Srinivasan], [Kumar-Saraf], [Kayal-Saha], [Kumar-Saraf], [Kayal-Saha], [Bera-Chakrabarti], [Kumar-Saraf], [Kumar-Saptharishi], [Forbes-Kumar-Saptharishi], [Kumar-Saraf],

Three types of improvements

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  Theorem ([Kayal-Saha-Saptharishi])

  An explicit polynomial f such that any depth-4 circuit* computing it must have size $n^{\Omega(\sqrt{d})}$.

  "Can we cross the $n^{\Omega(\sqrt{d})}$ threshold and get a $n^{\omega(\sqrt{d})}$ lower bound (and hence VP $\neq$ VNP)?"

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- **Lower bound for more general classes:**

  Theorem ([Kayal-Limaye-Saha-Srinivasan])
  An explicit polynomial f such that any hom. depth-4 circuit computing it must have size $n^{\Omega(\sqrt{d})}$.

  "Can we expect a more efficient depth reduction to this general class?"

Three types of improvements

Theorem

*An explicit polynomial f such that any depth-4 circuit** computing it must have size* $2^{\Omega(\sqrt{d})}$.

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  Theorem ([Kumar-Saraf])
  An explicit polynomial f, which is in VP, such that any depth-4 circuit* computing it must have size $n^{\Omega(\sqrt{d})}$.

  "Can we possibly hope to use this measure *as is* separate VP and VNP?"
  "Can we get a more efficient depth reduction? What about formulas, or depth-10 circuits?"

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... this talk is in this category

Our results

Theorem (Separation between depth-4 and 5)

There is an explicit n-variate degree $d = O(\log^2 n)$ polynomial f, that is computable by depth-5 circuits, such that any homogeneous depth-4 circuit computing it must have size $n^{\Omega(\sqrt{d})}$.

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So, no; you can't hope for a more efficient depth reduction to hom. depth-4 circuit from even depth-5 circuits.

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The polynomial happens to have a poly-sized non-hom. depth-3 circuit.

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Open: Get rid of the $d = O(\log^2 n)$ caveat.

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  Evaluations, derivatives, *shifts*, projections, etc.
- Somehow show that the measure is *small* for polynomials in $\mathscr{C}$, and *large* on the target polynomial.

Examples of complexity measures

- Hom. depth-3 circuits (or $\Sigma\Pi^{[d]}\Sigma$ circuits):

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Baby steps: What's the effect of a random restriction ρ on a depth-4 and a depth-5 circuit?

Effects of random restrictions

$$\rho \quad : \quad \text{set each } x_i \text{ to zero with probability } 1 - \frac{1}{n^{0.1}}$$

Depth-4 circuits: $\Sigma\Pi\Sigma\Pi$

Bottom level consists of monomials.

Effects of random restrictions

ρ : set each x_i to zero with probability $1 - \frac{1}{n^{0.1}}$

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Observation

Let $r = \sqrt{d}$. Then,

$$\Pr[\rho(x_1 \cdots x_r) \neq 0] \leq \frac{1}{n^{0.1r}}$$

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If a depth-4 circuit C has size s, then

$$\Pr[\rho(C) \text{ is not a bottom-support-} r \text{ circuit}] \quad \leq \quad \frac{s}{n^{0.1r}}$$

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Take away

If C is a small-ish $\Sigma\Pi\Sigma\Pi$ circuit, then $\rho(C)$ is a low-bottom-support depth-4 circuit whp.

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Depth-5 circuits: $\Sigma\Pi\Sigma\Pi\Sigma$

Bottom level consists of linear polynomials.

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Depth-5 circuits: $\Sigma\Pi\Sigma\Pi\Sigma$

Bottom level consists of linear polynomials.

Observation

If $\ell = x_1 + \cdots + x_m$, then

$$\Pr[\, \rho(\ell) = 0 \,] \quad \leq \quad \left(1 - \frac{1}{n^{0.1}}\right)^m$$

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Observation

If $\ell = x_1 + \cdots + x_m$ with $m = n^{0.2}$, then

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Hence, if C is a wide $\Sigma\Pi\Sigma\Pi\Sigma$ circuit of $\mathrm{poly}(n)$ size, then w.h.p. none of the bottom linear polynomials are killed.

Effects of random restrictions

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Take away

If C is a small-ish $\Sigma\Pi\Sigma\Pi$ circuit, then $\rho(C)$ is a low-bottom-support depth-4 circuit whp.

If C is a small and wide $\Sigma\Pi\Sigma\Pi\Sigma$ circuit, then $\rho(C)$ still has within it a high-bottom-support depth-4 circuit whp.

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[Kumar-Saraf] separated low-bottom-**degree** and high-bottom-**degree** depth-4 circuits.

Exploiting more structure

A generic $\Pi\Sigma\Pi$ circuit of high-bottom-support ($\approx 100\sqrt{d}$)

$$C \quad = \quad \prod_{i=1}^{\sqrt{d}/100} \sum_{j=1}^{m} \prod_{k=1}^{100\sqrt{d}} x_{ijk}$$

Observation

This polynomial is set-multilinear. That is, if

$$X_{i*k} \quad = \quad \left\{ x_{ijk} \,:\, j \in [m] \right\}$$

then every monomial in C contains exactly one variable in each X_{i*k}.

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Given a $P(X_1, \ldots, X_d)$, define the polynomial

$$P'(X_1, \ldots, X_d, y_1, \cdots, y_d) \quad = \quad P(y_1 X_1, \ldots, y_d X_d).$$

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$$\Gamma_{k,\ell}(P) \quad =$$

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Given a $P(X_1, \ldots, X_d)$, define the polynomial

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This measure is not new — a very similar used by [Kayal-Limaye-Saha-Srinivasan] to prove an $n^{\Omega(\sqrt{d})}$ lower bound for hom. depth-4 circuits computing IMM when $d = O(\log^2 n)$.

The proof

Lemma (Upper bound for circuit class)

If C is a size s hom. $\Sigma\Pi\Sigma\Pi$ circuit of low-bottom-support ($\leq \sqrt{d}$), then

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(The 2^{2d} term comes from a *set-multilinearization* step on C.) □

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If C is a generic wide (bottom degree $\approx 100\sqrt{d}$) $\Pi\Sigma\Pi$ circuit, then

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Here be dragons!

Get me out of here!

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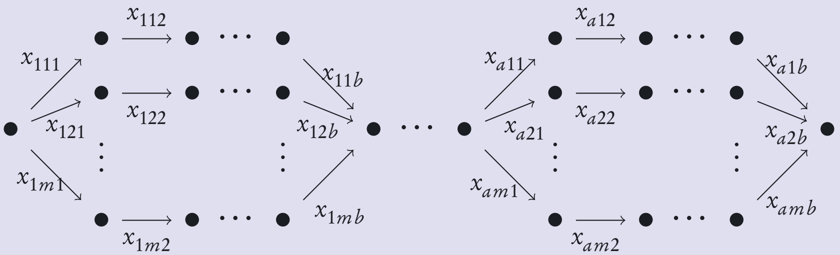

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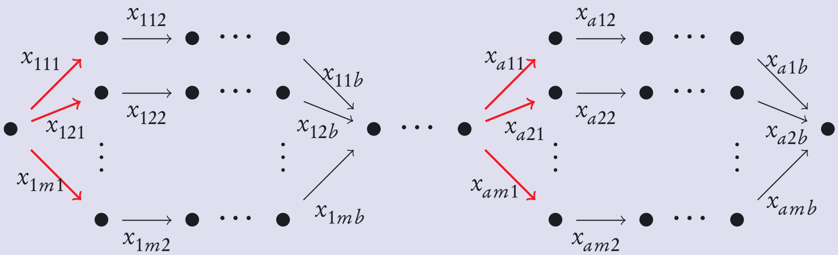

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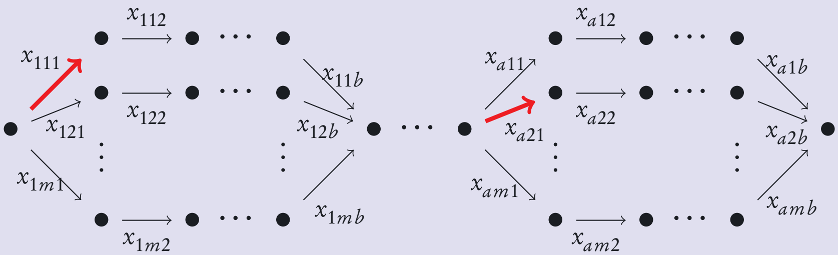

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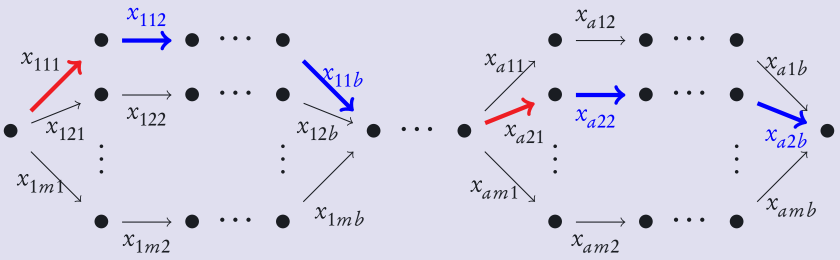

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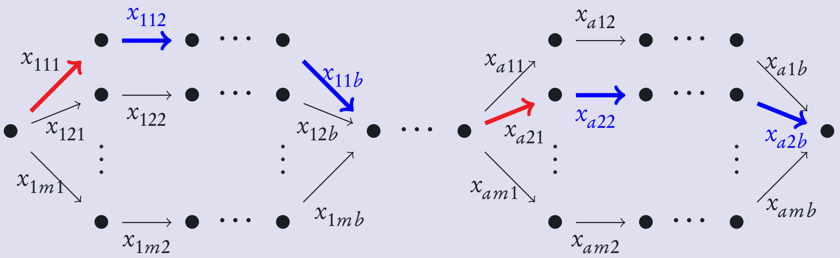

[Chillara-Mukhopadhyay]: If there are many *far* monomials, then there are many distinct shifts and hence $\Gamma_{k,\ell}(C)$ will be large.

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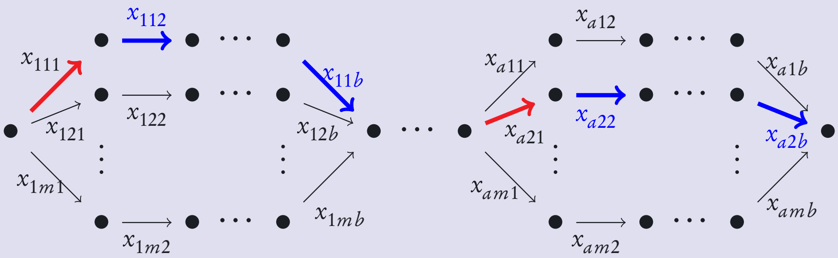

Observation. If derivatives are *far*, then so are the resulting monomials.

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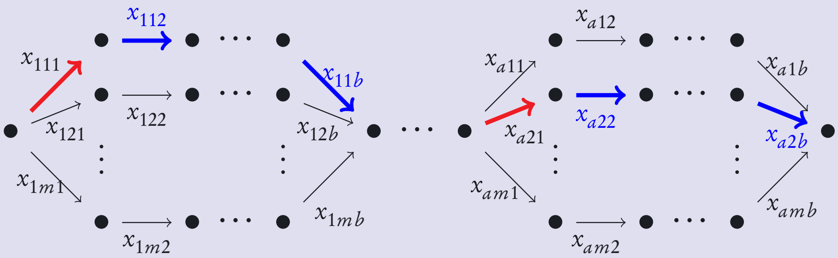

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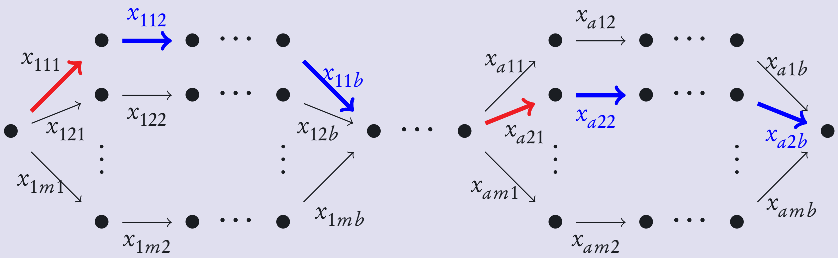

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