Lower bounds for shallow arithmetic circuits

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Tel Aviv University

Indian Institute of Technology Bombay,
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Complexity:

Can certain tasks

be computed

under certain resource constraints?

Time Complexity:

Can certain tasks

be computed

by polynomial time algorithms?

Space
Complexity:

Can certain tasks

be computed

by algorithms using just LOG-space?

Communication Complexity:

Can a boolean function $f(\mathbf{x}, \mathbf{y})$

be jointly computed

using very few bits of communication?

Circuit
Complexity:

Can a boolean function $f(\mathbf{x})$

be computed

by polynomial sized boolean circuits?
(made of AND, OR and NOT gates)

Arithmetic Circuit Complexity:

Can a polynomial $f(\mathbf{x})$

be computed

by polynomial sized arithmetic circuits?

(made of $+$ and $\times$ gates)

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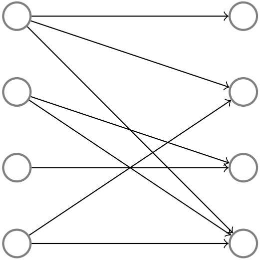
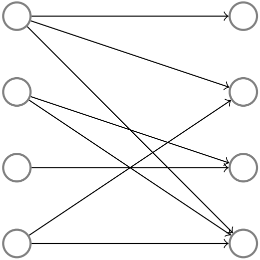
Focus of this talk

An application

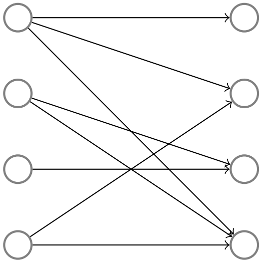


An application



Does there exist a perfect matching?

An application



Does there exist a perfect matching?

Want *efficient parallel* algorithms.

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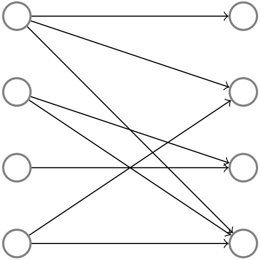

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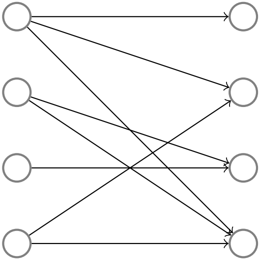
Tutte's Theorem

The graph has a perfect matching *if and only if*

$$\begin{vmatrix} x_{11} & x_{12} & 0 & x_{14} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & x_{33} & 0 \\ 0 & x_{42} & 0 & x_{44} \end{vmatrix} \neq 0$$

as a formal polynomial.

An application



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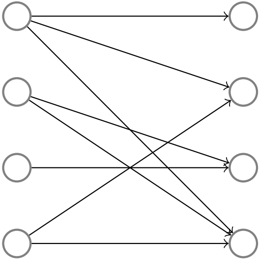

Does there exist a perfect matching?

Want *efficient parallel* algorithms.

Question: Can we test non-zeroness of "efficient polynomials"?

Firstly, what *are* efficient polynomials?

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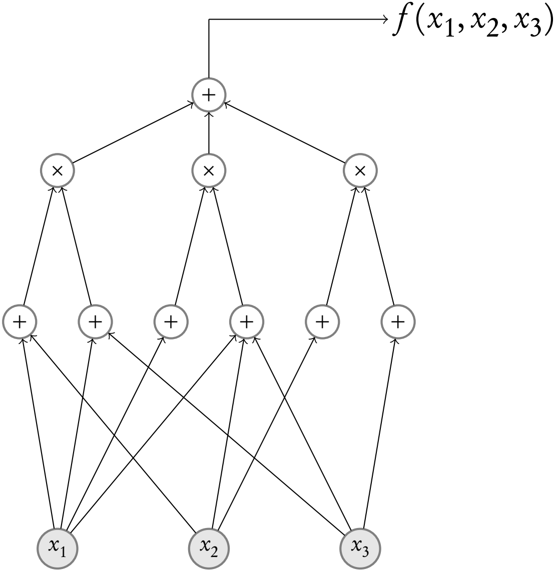

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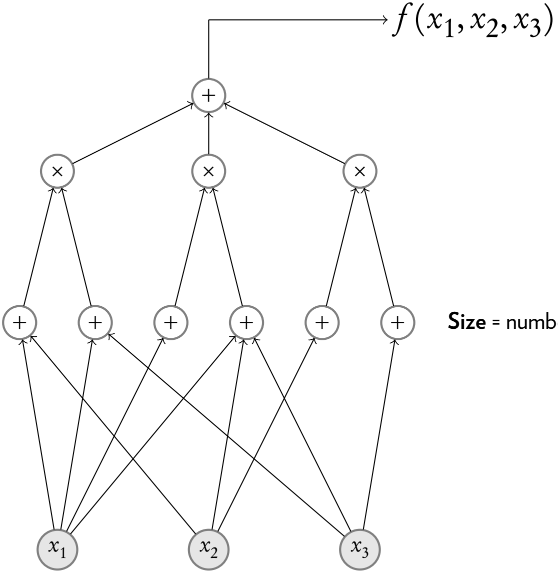
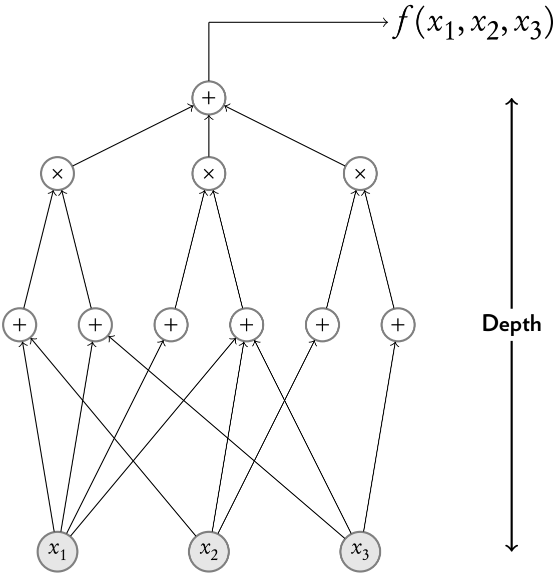


Size = number of gates

Arithmetic Circuits



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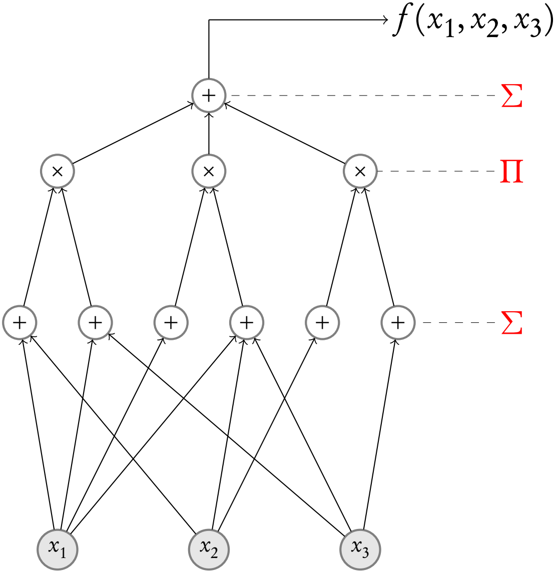

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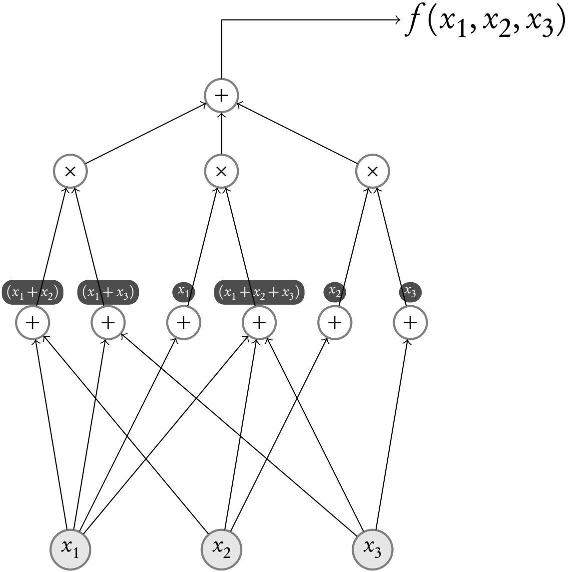

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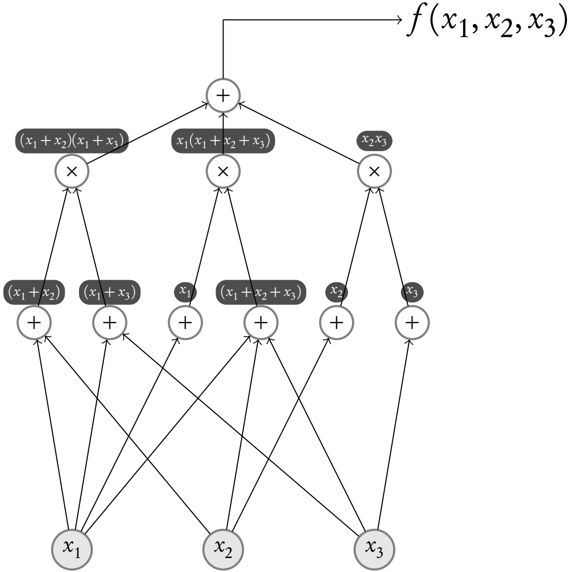

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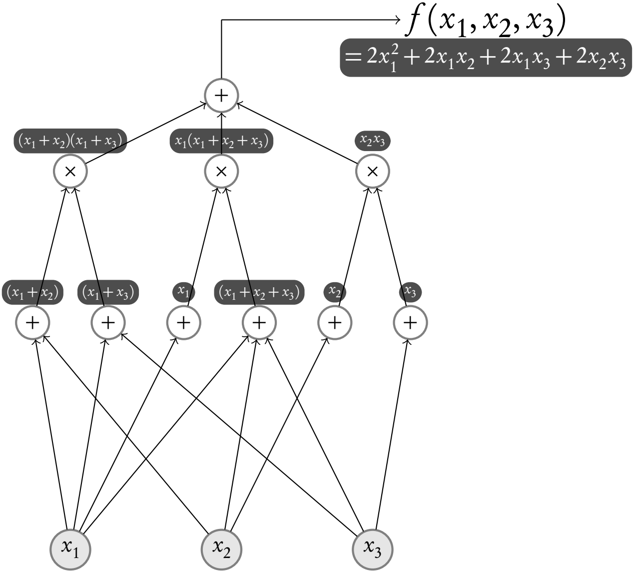

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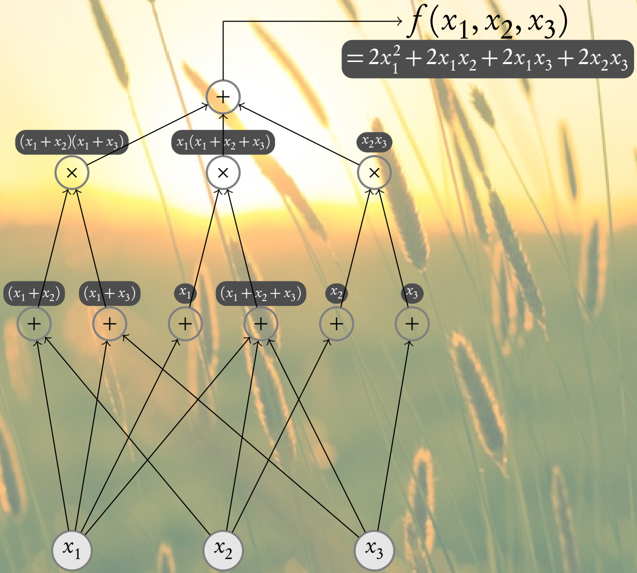

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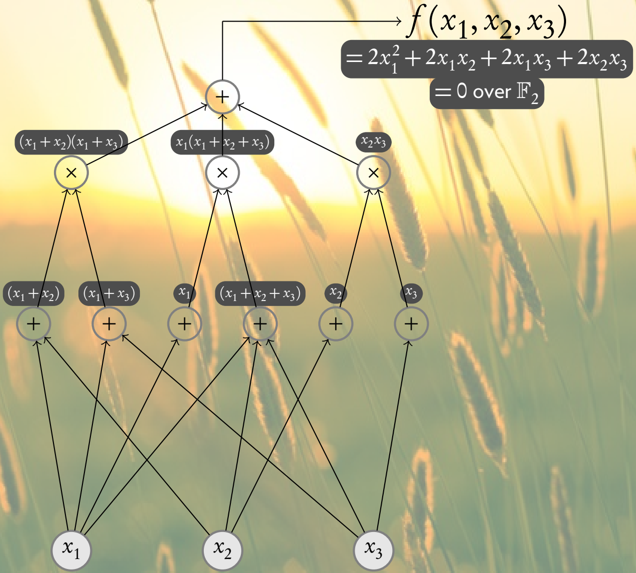

Algebraic analogue of P

Definition (Valiant's P, or efficient computation)

Polynomials $f(x_1, \ldots, x_n)$ that can be computed by $\mathrm{poly}(n)$-sized arithmetic circuits?

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Examples:

$$\text{[Ben-Or]} \quad \text{ESym}_d(x_1, \cdots, x_n) = \sum_{S \subseteq [n], |S| = d} \prod_{i \in S} x_i$$

$$\text{[Berkowitz,Mahajan-Vinay]} \quad \text{Det}_n = \begin{vmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}$$

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Fact: [Valiant] Det_n is complete* for VP.

Definition (Valiant's NP, or "explicit polynomials")

"Anything that can be succinctly described"

Examples:

$$\begin{aligned} \text{Perm}_n &= \operatorname{perm}\begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} \\ &= \sum_{\pi \in S_n} \prod_{i=1}^{n} x_{i\,\pi(i)} \end{aligned}$$

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- Given a monomial, the coefficient can be described easily.
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VP vs VNP

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$$\boxed{\text{VP} \quad \text{vs} \quad \text{VNP} \quad \overset{\sim}{\Longleftrightarrow} \quad \text{Det} \quad \text{vs} \quad \text{Perm}}$$

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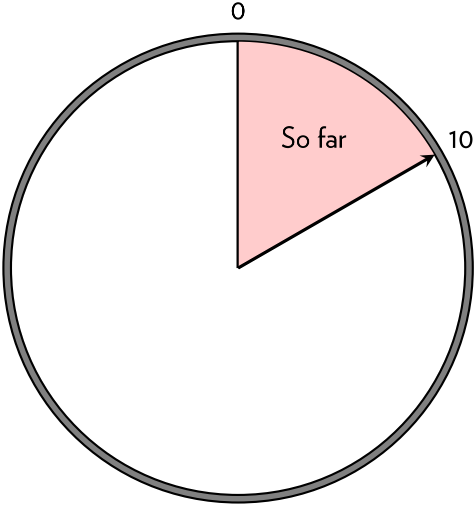
Math. The "Det vs Perm" is a very elegant mathematical question.

Fame *"The determinant of this conjecture would become permanently famous."*
– Neeraj Kayal

Outline



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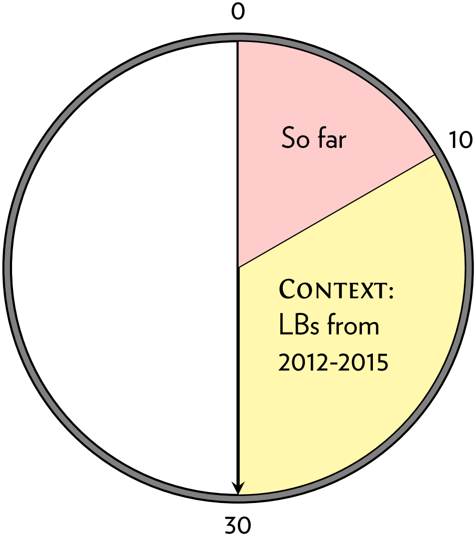

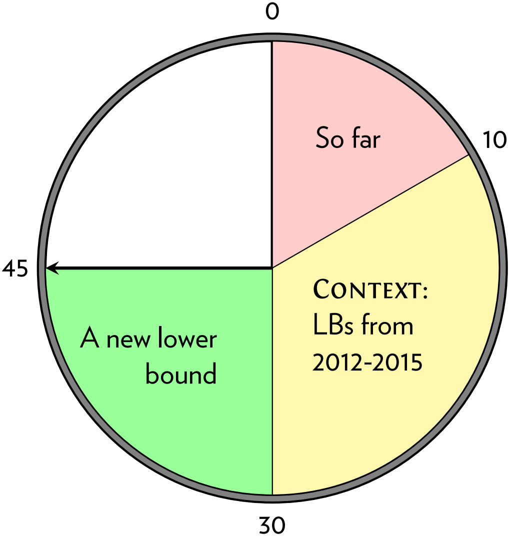
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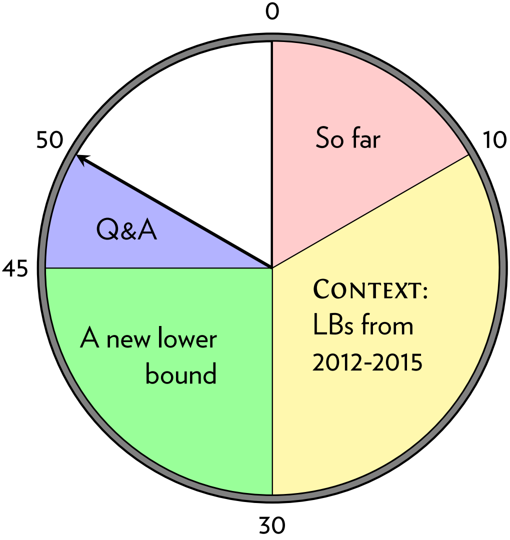
So far
10
45
Context:
LBs from
2012-2015
A new lower
bound
30

Outline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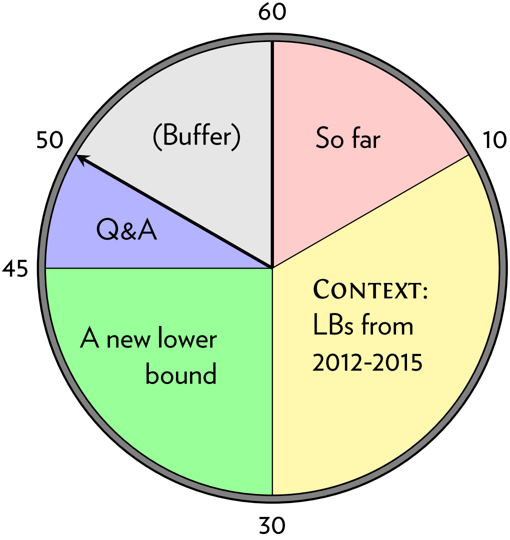
Outline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(Buffer)

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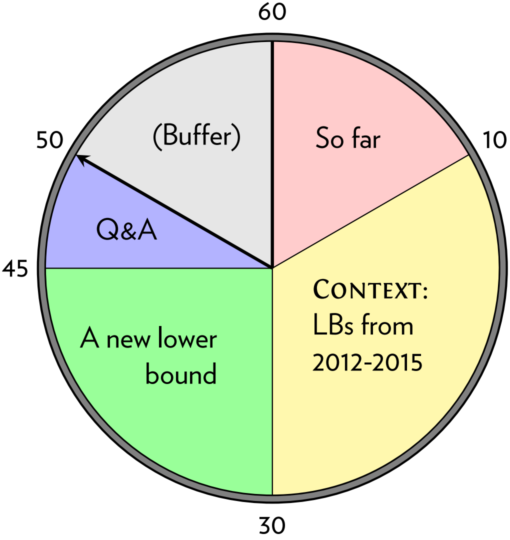

Diagram not to scale

Proof strategies to separate VP and VNP

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... lower bounds for small-depth circuits.

$\Sigma\Pi$ circuits

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- **DEPTH-2 CIRCUITS:**
  Sum of few monomials

$\Sigma\Pi$ circuits

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- **Depth-4 circuits:**
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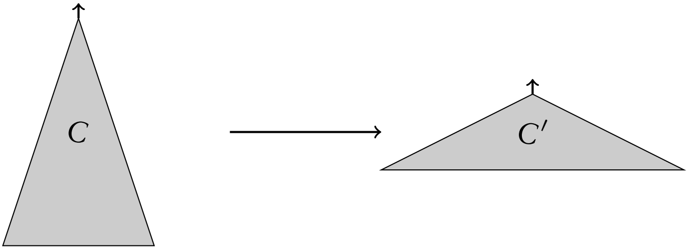
How powerful are such shallow circuits?

Depth reduction

Generic depth reduction

Can be computed by		Can be computed by
arithmetic circuits	⟶	“shallow” circuits
of “small” size		of “smallish” size



Depth reduction

Theorem ([Valiant-Skyum-Berkowitz-Rackoff-83])

Can be computed by		*Can be computed by*
arithmetic circuits	$\longrightarrow$	*log-depth circuits*
of $\mathrm{poly}(n,d)$ *size*		*of* $\mathrm{poly}(n,d)$ *size*

Reduction to depth-4

Theorem ([Agrawal-Vinay-08, Koiran-12, Tavenas-13])

Can be computed by		*Can be computed by*
arithmetic circuits	$\longrightarrow$	*depth-4 circuits*
of "small" size		*of "not-too-large" size*

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Depth-4 circuits* : $\sum \prod^{\sqrt{d}} \sum \prod^{\sqrt{d}}$ circuits

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Can be computed by		*Can be computed by*
arithmetic circuits	$\longrightarrow$	$\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$ *circuits*
of $\mathrm{poly}(n, d)$ *size*		*of* $n^{O(\sqrt{d})}$ *size*

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Cannot be computed by		*Cannot be computed by*
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of $\mathrm{poly}(n,d)$ *size*		*of* $n^{O(\sqrt{d})}$ *size*

Chasm at depth-4

Theorem ([Agrawal-Vinay-08, Koiran-12, Tavenas-13])

Can be computed by arithmetic circuits of $\text{poly}(n,d)$ size	$\longrightarrow$	Can be computed by $\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$ circuits of $n^{O(\sqrt{d})}$ size

(Or)

Cannot be computed by arithmetic circuits of $\text{poly}(n,d)$ size	$\longleftarrow$	**Cannot** be computed by $\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$ circuits of $n^{O(\sqrt{d})}$ size

The first crack in the dam...

Goal: Prove a good enough lower bound for $\sum\prod^{\sqrt{d}}\sum\prod^{\sqrt{d}}$

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An $2^{\Omega(d)}$ lower bound for $\Sigma\Pi^{d}\Sigma\Pi^{1}$ circuits.

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Theorem ([Kayal-12])

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Theorem ([Gupta-Kamath-Kayal-Saptharishi-12])

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Theorem ([Kayal-12])

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Theorem ([Gupta-Kamath-Kayal-Saptharishi-12])

An $2^{\Omega(\sqrt{d})}$ lower bound for $\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$ circuits.

... and the deluge that followed

	Lower bound	Circuit class	Polynomial
[GKK_1S_0-12]	$2^{\Omega(\sqrt{d})}$	$\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$	VP

G - Ankit Gupta
S_1 - Chandan Saha
S_2 - Srikanth Srinivasan
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... *another crack in the dam* ...

Theorem ([Agrawal-Vinay-08, Koiran-12, Tavenas-13])

Can be computed by		Can be computed by
arithmetic circuits	$\longrightarrow$	*depth-4 circuits**
of $\mathrm{poly}(n, d)$ *size*		*of* $n^{O(\sqrt{d})}$ *size*

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Surprising because

- such a result not true over small fields [Grigoriev-Karpinski-98],
- such a result not true for $\Sigma\Pi^{d}\Sigma$ circuits,
- no $\Sigma\Pi\Sigma$ circuit for Det_d was known better than size $d! = d^{O(d)}$ over any field.

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Another Goal: Prove an $n^{\omega(\sqrt{d})}$ lower bound for $\Sigma\Pi\Sigma^{\sqrt{d}}$ circuits.

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- Two possible ways to prove VP $\neq$ VNP:

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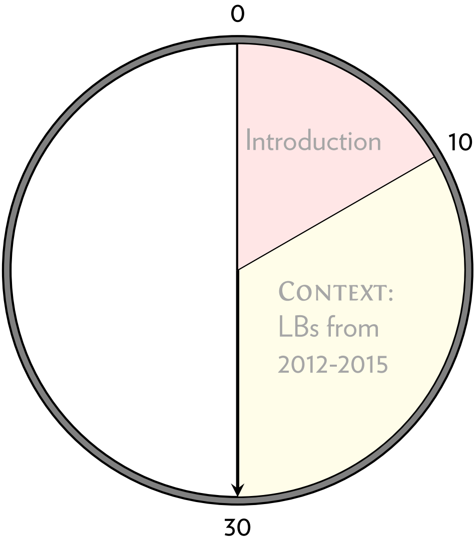
Self-plug: For those who want to know more details, here is a continuously updated survey: `http://github.com/dasarpmar/lowerbounds-survey/`

Outline



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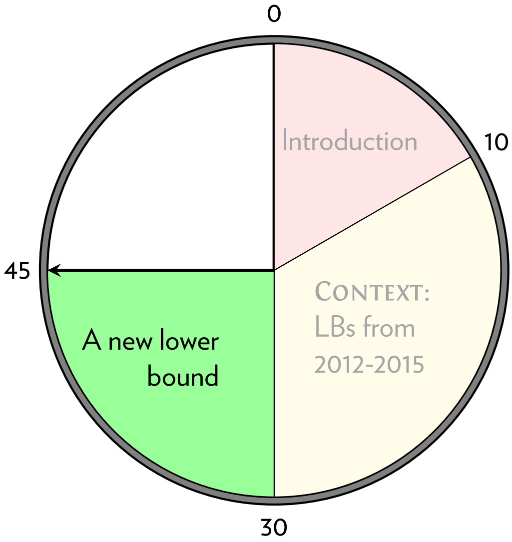
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Context:
LBs from
2012-2015
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A new lower
bound
30

Natural proof strategies

Construct a map $\Gamma : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{N}$, that assigns a number to every polynomial such that:

1. If f is computable by "small" circuits, then $\Gamma(f)$ is "small".
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Typically $\Gamma(f)$ is the rank of some associated linear space.

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&\in \operatorname{span}\left\{ \prod_{i \in S} \ell_i \; : \; S \subset [d] \, , \, |S| = d-1 \right\}
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  A generic polynomial is expected to have $n^{\Omega(d)}$ independent partial derivatives.

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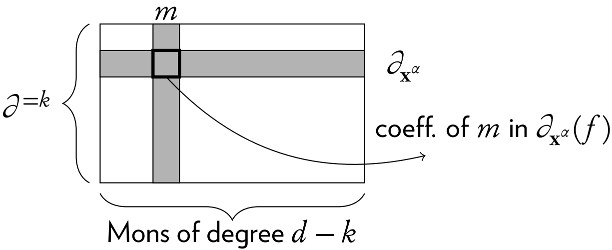

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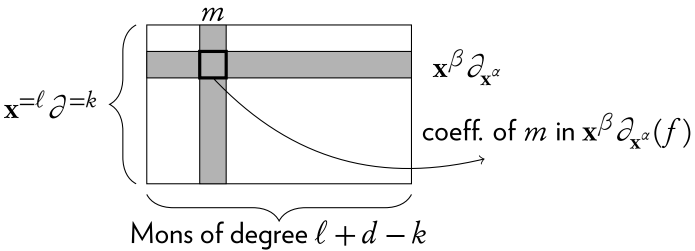

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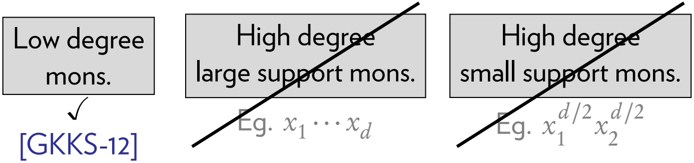

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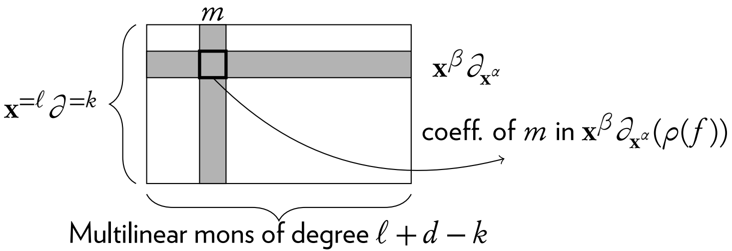

Moving to depth five

$$\Sigma\Pi\Sigma\Pi\Sigma$$

Sums of products of depth-3 circuits

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- **Answer:** The rank.

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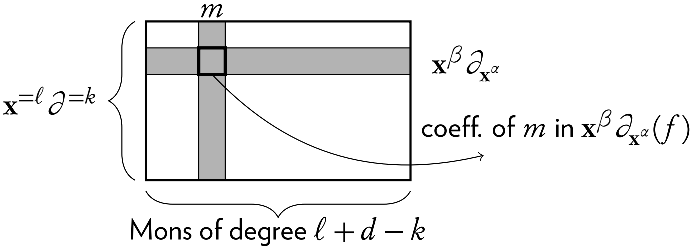

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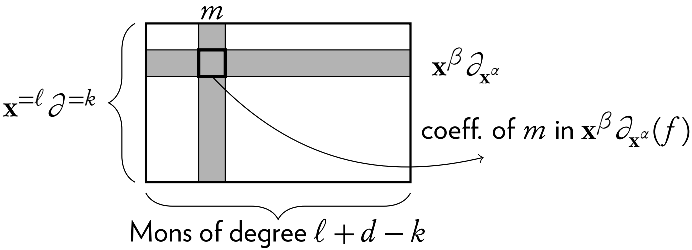
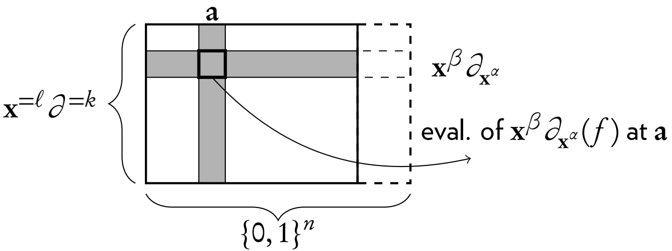


What we need now:



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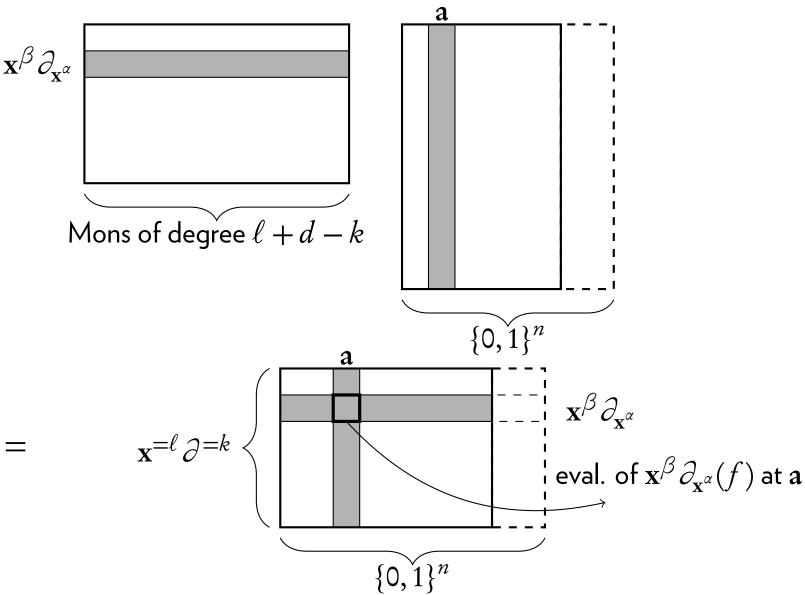

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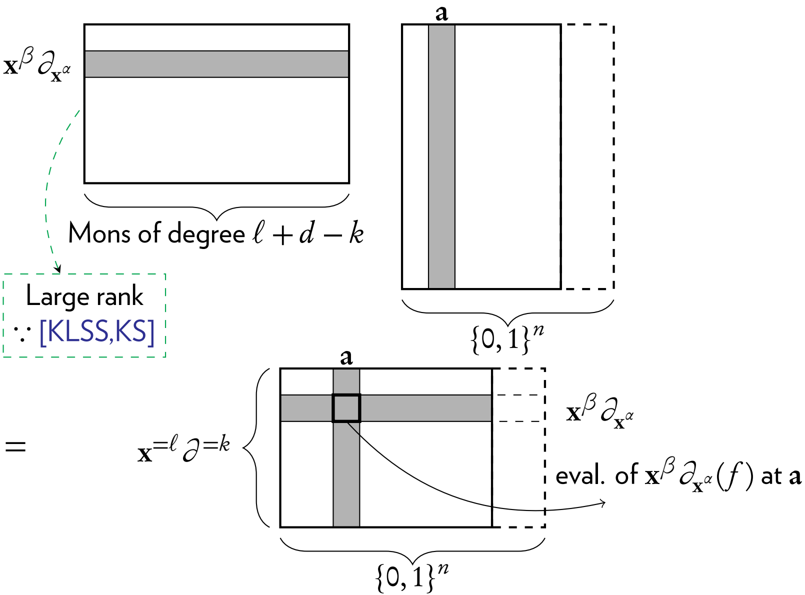

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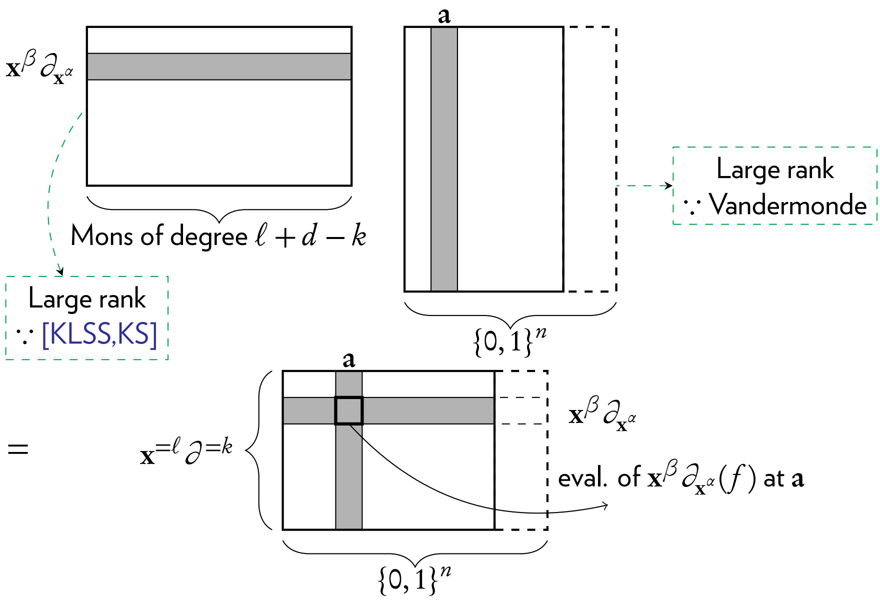

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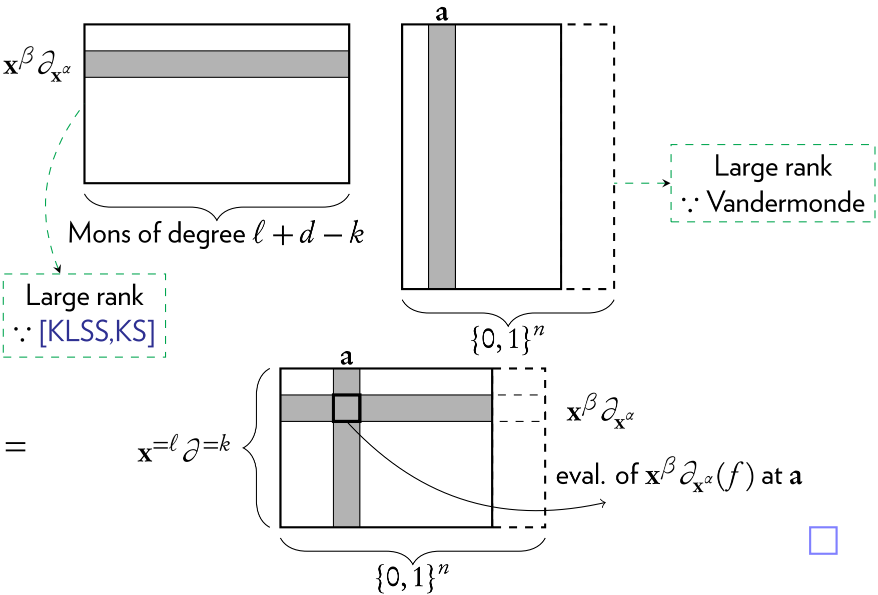

A lower bound

Theorem ([Kumar-Saptharishi-15])

There is a polynomial $f \in$ VNP such that, for every finite field $\mathbb{F}_q$, any hom. $\Sigma\Pi\Sigma\Pi\Sigma$ circuit computing f over $\mathbb{F}_q$ must have size $\exp(\Omega_q(\sqrt{d}))$.

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- Other fields?

Summary

"I lost you a while back... what do I need to remember?"

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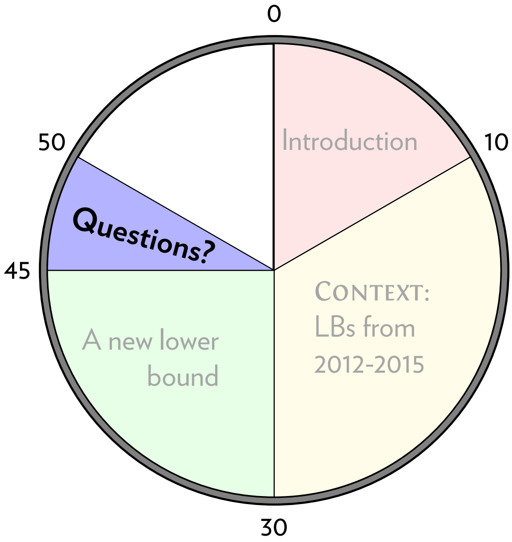
Thank you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Introduction
CONTEXT:
LBs from
2012-2015
A new lower
bound
Questions?