Shallow Water Equations: Flux-Based Wave Decomposition Solver

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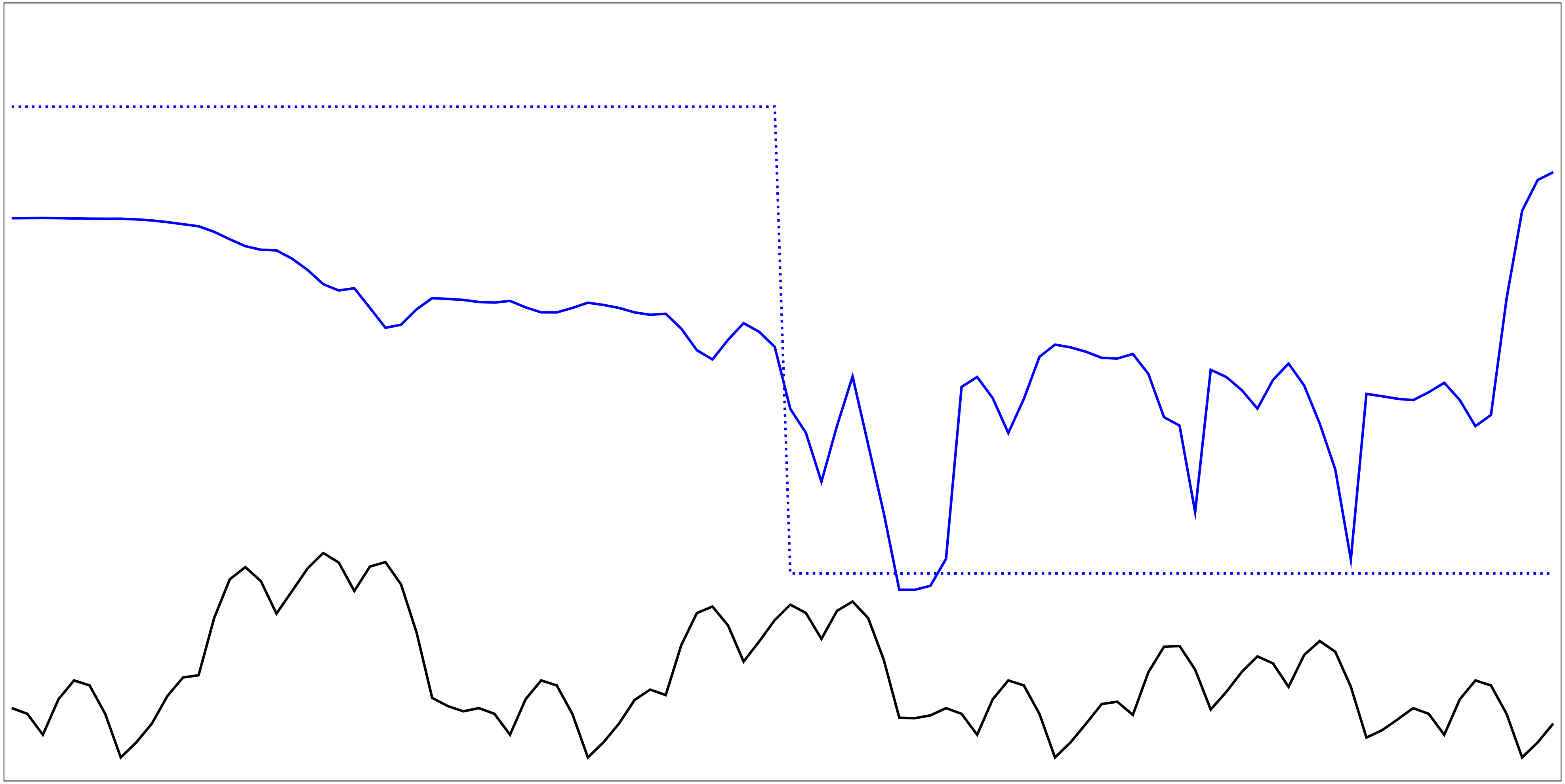

The Linear Riemann Problem

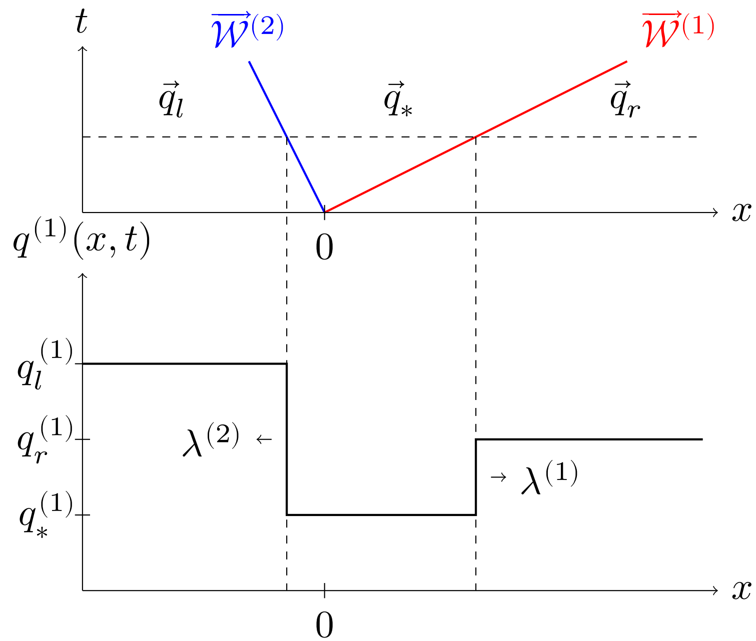

Figure 1 An example of the solution to the linear Riemann problem for $m = 2$

$$\vec{q}(x,t) = \vec{q}_l + \sum_{p:\ \lambda^{(p)} < x/t} \overrightarrow{\mathcal{W}}^{(p)} \tag{1}$$

$$\vec{q}(x,t) = \vec{q}_r - \sum_{p:\ \lambda^{(p)} > x/t} \overrightarrow{\mathcal{W}}^{(p)} \tag{2}$$

Godunov's Method

$$\vec{Q}_i^{n+1} = \vec{Q}_i^n - \frac{\Delta t}{\Delta x} \cdot \left(\vec{f}(\vec{Q}_{i+1/2}^n) - \vec{f}(\vec{Q}_{i-1/2}^n) \right), \quad \textbf{(3)}$$

where $\vec{Q}_{i\pm 1/2}^n$ are the solutions to the left and right Riemann problems (respective to the i-th grid cell) at the corresponding grid cell interfaces $x_{i\pm 1/2}$.

Key Facts to Consider

- nonlinear hyperbolic PDE $\Leftrightarrow$ nonlinear Riemann problem;
- full structure of the nonlinear Riemann problem (transonic waves) = expensive;
- little information (just one point) is needed in the end anyway.

What to do?

Seek for approximate solutions $\hat{\vec{Q}}_{i\pm 1/2}^n$ to the nonlinear Riemann problems.

How?

Linearize the Riemann problems locally by *quasilinear form* of differential conservation law:

$$\frac{\partial \vec{q}}{\partial t} + \vec{f}'(\vec{q}) \cdot \frac{\partial \vec{q}}{\partial x} = \vec{0}. \quad \textbf{(4)}$$

The Roe Solver: Idea

Solve the left and right linearized Riemann problems for $\vec{\alpha}_{i\pm1/2}$ to determine waves $\overrightarrow{\mathcal{W}}^{(p)}_{i\pm1/2}$:

$$\overrightarrow{Q}_i - \overrightarrow{Q}_{i-1} = \sum_{p=1}^{m} \alpha^{(p)}_{i-1/2} \cdot \hat{\vec{r}}^{(p)}_{i-1/2} = \sum_{p=1}^{m} \overrightarrow{\mathcal{W}}^{(p)}_{i-1/2}, \tag{5}$$

$$\overrightarrow{Q}_{i+1} - \overrightarrow{Q}_i = \sum_{p=1}^{m} \alpha^{(p)}_{i+1/2} \cdot \hat{\vec{r}}^{(p)}_{i+1/2} = \sum_{p=1}^{m} \overrightarrow{\mathcal{W}}^{(p)}_{i+1/2}. \tag{6}$$

Using **(1)** and **(2)** we can define

$$\vec{f}(\overrightarrow{Q}^n_{i-1/2}) = \vec{f}(\overrightarrow{Q}_i) - \sum_{p=1}^{m} (\hat{\lambda}^{(p)}_{i-1/2})^+ \cdot \overrightarrow{\mathcal{W}}^{(p)}_{i-1/2}, \tag{7}$$

$$\vec{f}(\overrightarrow{Q}^n_{i+1/2}) = \vec{f}(\overrightarrow{Q}_i) + \sum_{p=1}^{m} (\hat{\lambda}^{(p)}_{i+1/2})^- \cdot \overrightarrow{\mathcal{W}}^{(p)}_{i+1/2}, \tag{8}$$

where waves $\overrightarrow{\mathcal{W}}^{(p)}_{i\pm1/2}$ are propagating at the corresponding constant speeds $\hat{\lambda}^{(p)}_{i\pm1/2}$, which are eigenvalues of averaged Jacobians $\hat{A}_{i\pm1/2}$ respectively.

Plugging **(7)** and **(8)** into **(3)** we arrive to the classical Godunov's method implementation:

$$\overrightarrow{Q}^{n+1}_i = \overrightarrow{Q}^n_i - \frac{\Delta t}{\Delta x} \cdot \left(\sum_{p=1}^{m} (\hat{\lambda}^{(p)}_{i+1/2})^- \cdot \overrightarrow{\mathcal{W}}^{(p)}_{i+1/2} + \sum_{p=1}^{m} (\hat{\lambda}^{(p)}_{i-1/2})^+ \cdot \overrightarrow{\mathcal{W}}^{(p)}_{i-1/2} \right). \tag{9}$$

The Roe Solver: Conservation

Is (9) conservative with any choice of $\hat{A}_{i\pm1/2}$, if waves are calculated by (5) and (6)?

No.

What does it mean to be conservative?

$$\vec{f}(\overrightarrow{Q}_i) - \vec{f}(\overrightarrow{Q}_{i-1}) = \sum_{p=1}^{m} \hat{\lambda}^{(p)}_{i-1/2} \cdot \overrightarrow{\mathcal{W}}^{(p)}_{i-1/2} \tag{10}$$

$$\vec{f}(\overrightarrow{Q}_{i+1}) - \vec{f}(\overrightarrow{Q}_i) = \sum_{p=1}^{m} \hat{\lambda}^{(p)}_{i+1/2} \cdot \overrightarrow{\mathcal{W}}^{(p)}_{i+1/2} \tag{11}$$

So how to make (9) conservative?

Utilizing **(5)** and **(6)**, we can immediately rewrite **(10)** and **(11)** as

$$\vec{f}(\overrightarrow{Q}_i) - \vec{f}(\overrightarrow{Q}_{i-1}) = \hat{A}_{i-1/2} \cdot (\overrightarrow{Q}_i - \overrightarrow{Q}_{i-1}), \tag{12}$$

$$\vec{f}(\overrightarrow{Q}_{i+1}) - \vec{f}(\overrightarrow{Q}_i) = \hat{A}_{i+1/2} \cdot (\overrightarrow{Q}_{i+1} - \overrightarrow{Q}_i), \tag{13}$$

what gives us the restrictions that have to be imposed on the averaged Jacobians $\hat{A}_{i\pm1/2}$ so that **(9)** is guaranteed to be conservative.

The Roe Solver: Consequences

Substantial amount of effort has been put into defining Roe averaged Jacobians having property **(12)** and **(13)** for various nonlinear problems such as *shallow water equations* or *Euler equations* for gas dynamics. However, for some problems Roe averaged Jacobians either cannot be easily computed or cannot be defined at all.

Summary of Disadvantages

- not generic and inflexible: requires special linearization of Jacobians (which might be not available for certain problems) in order to be conservative;
- unclear how to incorporate the *source term* properly.

Can we do better?

Yes!

The Flux-Based Wave Decomposition Solver: Idea

Recall conservation conditions **(10)** and **(11)**, and let's rewrite them as

$$\vec{f}(\overrightarrow{Q}_i) - \vec{f}(\overrightarrow{Q}_{i-1}) = \sum_{p=1}^{m} \hat{\lambda}_{i-1/2}^{(p)} \cdot \overrightarrow{\mathcal{W}}_{i-1/2}^{(p)} = \sum_{p=1}^{m} \overrightarrow{\mathcal{Z}}_{i-1/2}^{(p)}, \tag{14}$$

$$\vec{f}(\overrightarrow{Q}_{i+1}) - \vec{f}(\overrightarrow{Q}_{i}) = \sum_{p=1}^{m} \hat{\lambda}_{i+1/2}^{(p)} \cdot \overrightarrow{\mathcal{W}}_{i+1/2}^{(p)} = \sum_{p=1}^{m} \overrightarrow{\mathcal{Z}}_{i+1/2}^{(p)}, \tag{15}$$

where $\overrightarrow{\mathcal{Z}}_{i\pm1/2}^{(p)}$ are the so-called *flux waves*.

The important observation is that since $\overrightarrow{\mathcal{W}}_{i\pm1/2}^{(p)}$ are eigenvectors of averaged Jacobians $\hat{A}_{i\pm1/2}$, $\overrightarrow{\mathcal{Z}}_{i\pm1/2}^{(p)}$ are their eigenvectors too. As a result, it turns out that, instead of decomposing the jump in $\vec{q}$ (see **(5)** and **(6)**), we can directly decompose the jump in $\vec{f}(\vec{q})$ into the flux waves by solving

$$\vec{f}(\overrightarrow{Q}_i) - \vec{f}(\overrightarrow{Q}_{i-1}) = \sum_{p=1}^{m} \beta_{i-1/2}^{(p)} \cdot \hat{\vec{r}}_{i-1/2}^{(p)} = \sum_{p=1}^{m} \overrightarrow{\mathcal{Z}}_{i-1/2}^{(p)}, \tag{16}$$

$$\vec{f}(\overrightarrow{Q}_{i+1}) - \vec{f}(\overrightarrow{Q}_{i}) = \sum_{p=1}^{m} \beta_{i+1/2}^{(p)} \cdot \hat{\vec{r}}_{i+1/2}^{(p)} = \sum_{p=1}^{m} \overrightarrow{\mathcal{Z}}_{i+1/2}^{(p)} \tag{17}$$

for $\vec{\beta}_{i\pm1/2}$ to obtain $\overrightarrow{\mathcal{Z}}_{i\pm1/2}^{(p)}$.

The Flux-Based Wave Decomposition Solver: Conservation

Finally, if we define waves as

$$\overrightarrow{\mathcal{W}}^{(p)}_{i-1/2} = \frac{\overrightarrow{\mathcal{Z}}^{(p)}_{i-1/2}}{\hat{\lambda}^{(p)}_{i-1/2}}, \tag{18}$$

$$\overrightarrow{\mathcal{W}}^{(p)}_{i+1/2} = \frac{\overrightarrow{\mathcal{Z}}^{(p)}_{i+1/2}}{\hat{\lambda}^{(p)}_{i+1/2}}, \tag{19}$$

and plug **(18)** and **(19)** into **(9)**, we will obtain **conservative** finite volume method out-of-the-box.

Is (9) conservative with any choice of $\hat{A}_{i\pm 1/2}$,
if waves are calculated by (18) and (19)?

Indeed!

What is the benefit?

$$\hat{A}_{i-1/2} = \vec{f}'\left(\frac{\overrightarrow{Q}_i + \overrightarrow{Q}_{i-1}}{2}\right) \tag{20}$$

$$\hat{A}_{i+1/2} = \vec{f}'\left(\frac{\overrightarrow{Q}_{i+1} + \overrightarrow{Q}_i}{2}\right) \tag{21}$$

Of course explicitly using **(18)** and **(19)** with **(9)** in practice is not a good idea. The reason is that both **(18)** and **(19)** can obviously exhibit numerical instabilities when $\hat{\lambda}^{(p)}_{i\pm 1/2} \to 0$.

Much better approach is to turn **(9)** into the new update scheme which relies flux waves $\overrightarrow{\mathcal{Z}}^{(p)}_{i\pm 1/2}$ directly:

$$\overrightarrow{Q}^{n+1}_i = \overrightarrow{Q}^{n}_i - \frac{\Delta t}{\Delta x} \cdot \left(\sum_{p=1}^{m} \left| \mathrm{sgn}^{-}(\hat{\lambda}^{(p)}_{i+1/2}) \right| \cdot \overrightarrow{\mathcal{Z}}^{(p)}_{i+1/2} + \sum_{p=1}^{m} \left| \mathrm{sgn}^{+}(\hat{\lambda}^{(p)}_{i-1/2}) \right| \cdot \overrightarrow{\mathcal{Z}}^{(p)}_{i-1/2} \right), \qquad \textbf{(22)}$$

where

$$\mathrm{sgn}^{+}(\lambda) = \max(\mathrm{sgn}(\lambda), 0),$$

$$\mathrm{sgn}^{-}(\lambda) = \min(\mathrm{sgn}(\lambda), 0).$$

Balance Law, Source Term, and Bathymetry

The *differential balance law*

$$\frac{\partial}{\partial t}\vec{q}(x,t) + \frac{\partial}{\partial x}\vec{f}(\vec{q}(x,t)) = \vec{\psi}(\vec{q}(x,t),x) \tag{23}$$

consists of a *differential conservation law* with a source term $\vec{\psi}(\vec{q}(x,t),x)$ on the right-hand side.

Underwater *topography* is generally called *bathymetry* further denoted by $B(x)$. The free surface of the fluid is then given by

$$S(x,t) = h(x,t) + B(x). \tag{24}$$

Furthermore, $H(x)$ denotes the distance from bathymetry $B(x)$ to some constant *reference level* L of the fluid surface:

$$L = H(x) + B(x). \tag{25}$$

The shallow water equations then take the form

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ h\cdot u \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} h\cdot u \\ h\cdot u^2 + \frac{1}{2}\cdot g\cdot h^2 \end{pmatrix} = \begin{pmatrix} 0 \\ g\cdot h\cdot H'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -g\cdot h\cdot B'(x) \end{pmatrix}, \tag{26}$$

where, according to **(23)**,

$$\vec{\psi}(\vec{q},x) = \begin{pmatrix} 0 \\ -g\cdot q^{(1)}\cdot B'(x) \end{pmatrix}. \tag{27}$$

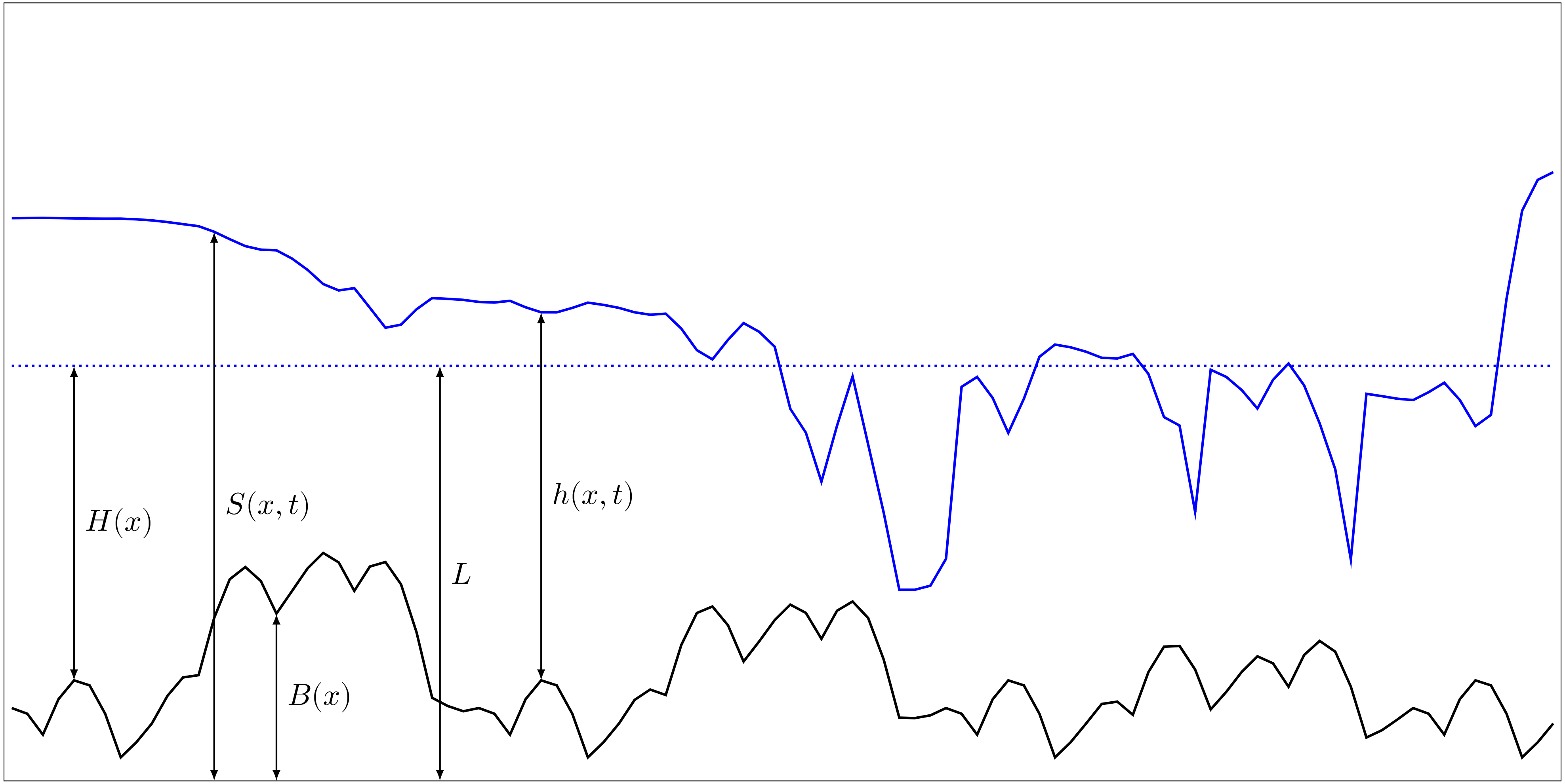

Figure 2 Shallow water equations with bathymetry

The Flux-Based Wave Decomposition Solver: Source Term Treatment

Which method is called "well-balanced"?

A method is called *well-balanced* if equilibrium initial data is exactly preserved by the method. Furthermore, the method should also accurately resolve solutions that are small deviations from equilibrium data.

Many numerical approaches have been studied to solve **(23)** properly. However, they have hard time dealing with the solutions that are close to equilibrium state, i.e. when

$$\frac{\partial}{\partial x}\vec{f}(\vec{q}(x,t)) \approx \vec{\psi}(\vec{q}(x,t),x), \tag{28}$$

while each term separately is large.

The flux-based wave decomposition approach can incorporate left and right source term averages $\overrightarrow{\Psi}_{i\pm 1/2}$ directly into **(16)** and **(17)** to yield

$$\vec{f}(\overrightarrow{Q}_i) - \vec{f}(\overrightarrow{Q}_{i-1}) - \Delta x \cdot \overrightarrow{\Psi}_{i-1/2} = \sum_{p=1}^{m} \overrightarrow{\mathcal{Z}}^{(p)}_{i-1/2}, \tag{29}$$

$$\vec{f}(\overrightarrow{Q}_{i+1}) - \vec{f}(\overrightarrow{Q}_i) - \Delta x \cdot \overrightarrow{\Psi}_{i+1/2} = \sum_{p=1}^{m} \overrightarrow{\mathcal{Z}}^{(p)}_{i+1/2}, \tag{30}$$

what makes the flux-based wave decomposition solver superior to others.

The Flux-Based Wave Decomposition Solver: Source Term Treatment

Why (29) and (30) are superior to other approaches?

In contrast to the other methods, this approach is particularly attractive in cases where the solution is close to the equilibrium state, i.e. when (28) takes place. To understand why, recall (28) and consider left and right discretized versions of it:

$$\frac{\vec{f}(\overrightarrow{Q}_i) - \vec{f}(\overrightarrow{Q}_{i-1})}{\Delta x} = \overrightarrow{\Psi}_{i-1/2}, \tag{31}$$

$$\frac{\vec{f}(\overrightarrow{Q}_{i+1}) - \vec{f}(\overrightarrow{Q}_i)}{\Delta x} = \overrightarrow{\Psi}_{i+1/2}. \tag{32}$$

The left-hand sides of both (29) and (30) will be zero respectively, and hence all the flux waves $\overrightarrow{\mathcal{Z}}^{(p)}_{i\pm 1/2}$ will have zero strength, which is the indication of numerical equilibrium state satisfying (31) and (32) being maintained exactly. As a result, it turns out that the method can be **well-balanced**. The only trick is to choose an appropriate averaging scheme for the source term.

The Flux-Based Wave Decomposition Solver: Source Term Treatment

So how to define averaged source terms $\overrightarrow{\Psi}_{i\pm1/2}$ properly for shallow water equations?

The so-called *surface-at-rest* is a very important equilibrium case, and it can be expressed as

$$\begin{cases} u_e \equiv 0 \\ h_e(x) + B(x) = S_e \equiv \text{const} \end{cases}. \quad \textbf{(33)}$$

The flux-based wave decomposition method is **well-balanced** if left and right source term averages $\overrightarrow{\Psi}_{i\pm1/2}$ are chosen as follows:

$$\overrightarrow{\Psi}_{i-1/2} = \begin{pmatrix} 0 \\ -g \cdot \dfrac{h_i + h_{i-1}}{2} \cdot \dfrac{B_i - B_{i-1}}{\Delta x} \end{pmatrix}, \quad \textbf{(34)}$$

$$\overrightarrow{\Psi}_{i+1/2} = \begin{pmatrix} 0 \\ -g \cdot \dfrac{h_{i+1} + h_i}{2} \cdot \dfrac{B_{i+1} - B_i}{\Delta x} \end{pmatrix}. \quad \textbf{(35)}$$

Why?

It can be verified by direct substitution that when **(33)** takes place and source terms are chosen as **(34)** and **(35)**, then both **(31)** and **(32)** are satisfied exactly.

The Flux-Based Wave Decomposition Solver: Conclusion

Summary of Advantages

- generic and flexible: can be applied to a wide variety of problems (including those where Roe averaged Jacobians are not available);
- source term is incorporated naturally yielding well-balanced numerical scheme (when source term averages are defined properly);
- naturally extends to spatially varying fluxes (something that was not covered, but something to keep in mind).

Finally, the flux-based wave decomposition solver for the shallow water equations is extensively used in tsunami simulation, an application where it is particularly critical that the method is **well-balanced**, so that small perturbations around the *ocean-at-rest* state are accurately captured since the magnitude of a tsunami wave is generally one meter or even less while the bathymetry varies on the order of several kilometers.