

Depth reduction in arithmetic circuits

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WACT
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Polynomials

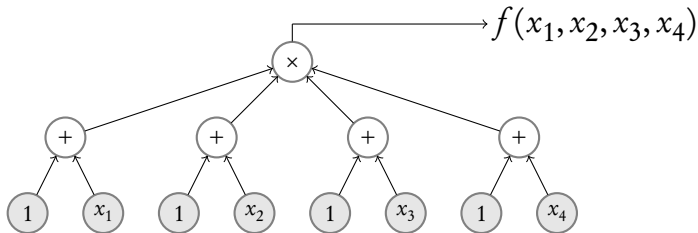
$$\begin{aligned} f(x_1, x_2, x_3, x_4) = & 1 + x_1 + x_2 + x_3 + x_4 \\ & + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\ & + x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 + x_1x_2x_3 \\ & + x_1x_2x_3x_4 \end{aligned}$$

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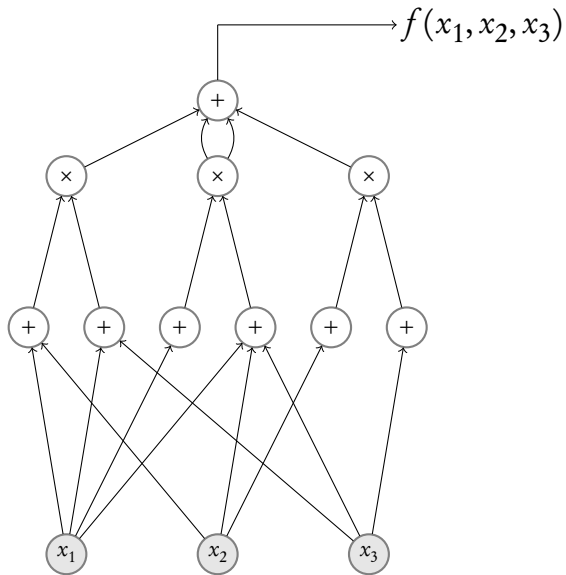
“How hard is it to compute a given n -variate degree d polynomial?”

Arithmetic Formulae

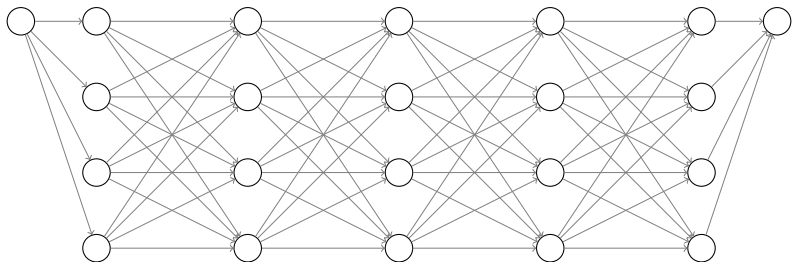


- ▶ Tree
- ▶ Leaves containing variables or constants

Arithmetic Circuits

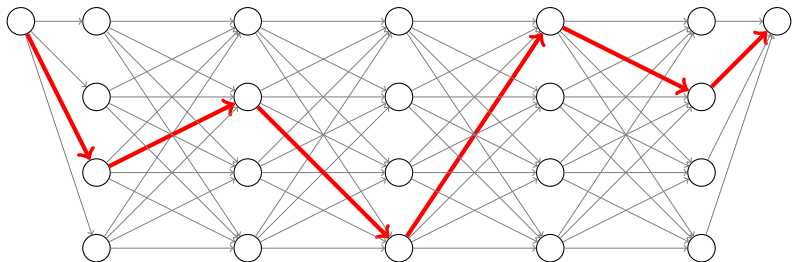


Arithmetic Branching Programs



A directed layered graph with a unique source node s and sink node t .
Each edge e holds a linear polynomial ℓ_e .

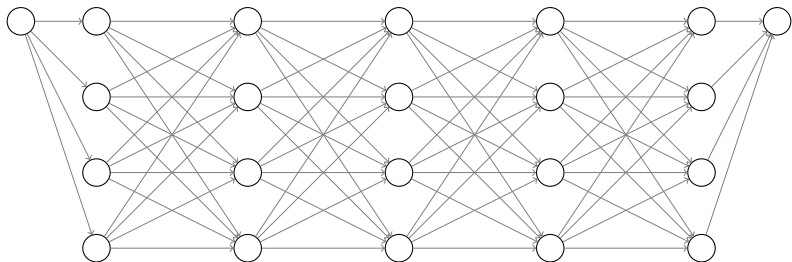
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$$\text{wt}(P) = \prod_{e \in P} \ell_e$$

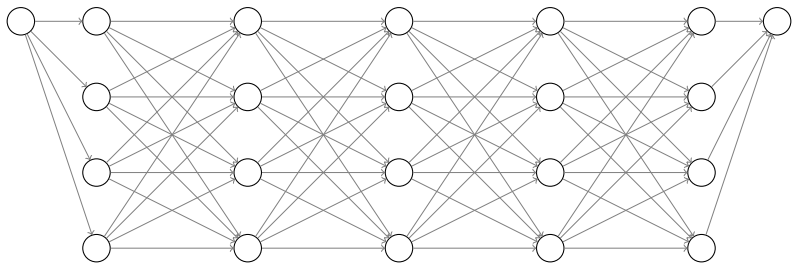
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Equivalent to iterated matrix product

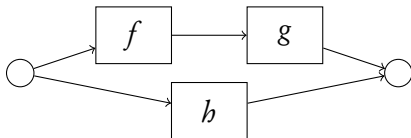
Relationship between these classes

Formulas \subseteq ABP

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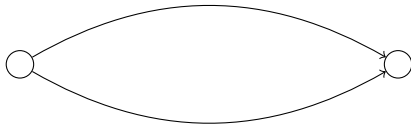
Formulas \subseteq ABP

$fg + b$:



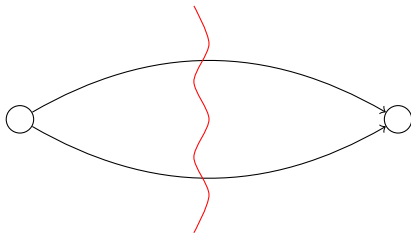
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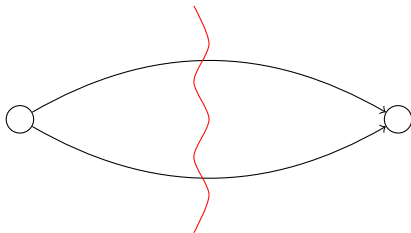
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Savitch

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$$g = g_1 + g_2 \longrightarrow g^{(i)} = g_1^{(i)} + g_2^{(i)}$$

$$g = g_1 \times g_2 \longrightarrow g^{(i)} = \sum_{j=0}^i g_1^{(j)} \times g_2^{(i-j)}$$

The illustrious siblings

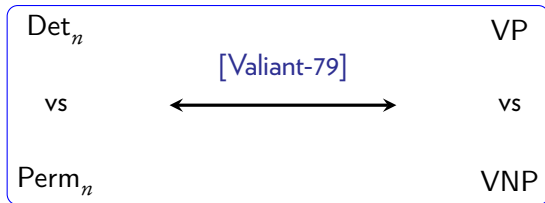
$$\text{Det}_n(x_{11}, \dots, x_{nn}) = \sum_{\sigma \in \mathcal{S}_n} \text{sign}(\sigma) \cdot x_{1\sigma(1)} \cdots x_{n\sigma(n)}$$

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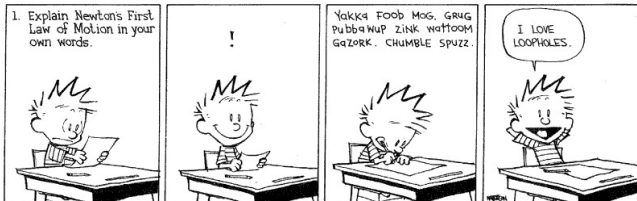
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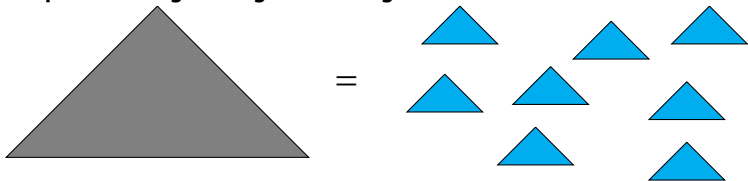
General roadmap for lower bounds

Four steps in most lower bound proofs

Step 1: Finding the right building blocks

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Meta Theorem 1

Every **small** circuit can be equivalently computed as a *sum* of **few** s

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Meta Theorem 2

Find a map $\Gamma : \mathbb{F}[\mathbf{x}] \rightarrow \mathbb{Z}_{\geq 0}$ such that $\Gamma(\text{triangle})$ is **small**.

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Convince yourself that $\Gamma(R)$ *must be* **LARGE** for a random polynomial R .

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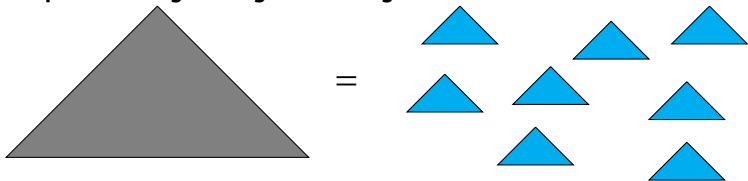
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Step 4: Find a hay in the haystack

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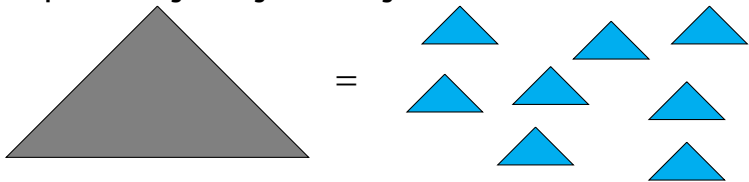


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Depth reduction

A short history of depth reduction

Circuit Class	Depth	Size	
Formulas	$O(\log s)$	$\text{poly}(s)$	[Brent]

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Circuits	3^*	$s^{O(\sqrt{d})}$	[Gupta-Kamath-Kayal-S]

Other depth reductions in lower bounds

Multilinear formulas

[Raz, Raz-Yehudayoff]

$$f = \sum_{i=1}^s g_{i1} \cdot g_{i2} \cdots g_{il} \quad , \quad (1/3)^j \cdot n \leq \text{Var}(g_{ij}) \leq (2/3)^j \cdot n$$

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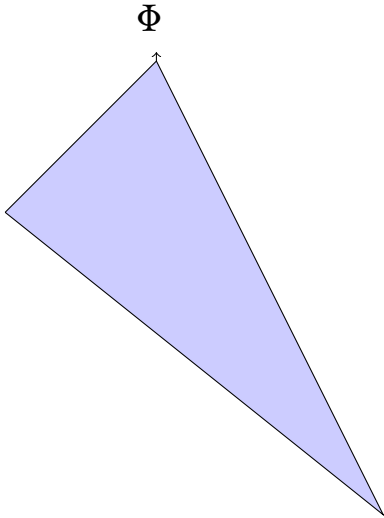
Homogeneous $\Sigma\Pi\Sigma\Pi$

[Kayal-Limaye-Saha-Srinivasan]

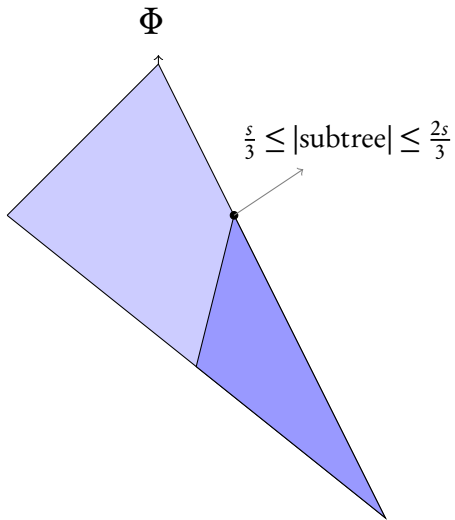
$$f = \Sigma\Pi\Sigma\Pi^{[\sqrt{d}]} + \sum_{i=1}^s m_i Q_i \quad , \quad \text{deg}(m_i) \geq \sqrt{d}$$

- ▶ Classical depth reductions of [Brent] and [VSBR].
- ▶ A *slightly* different proof of [Tavenas]
- ▶ Better building blocks for homogeneous formulas
- ▶ (depending on time) Reduction to depth three [GKS2]

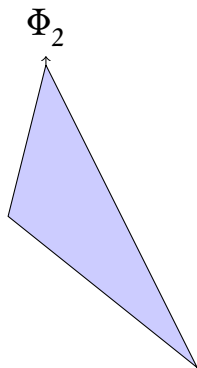
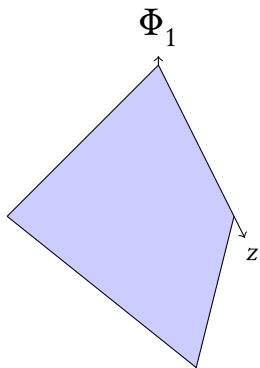
Depth reducing formulas



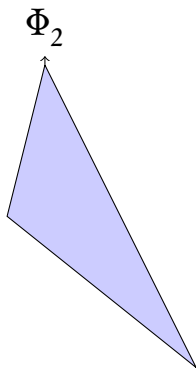
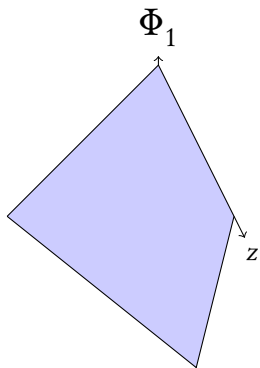
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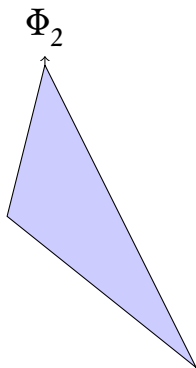
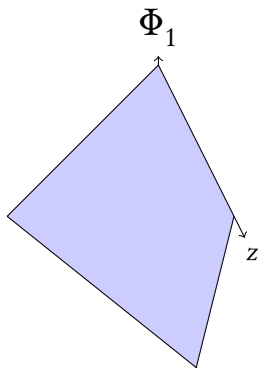


Depth reducing formulas



$$\begin{aligned}\Phi_1(z) &= A \cdot z + B \\ \Phi &= A \cdot \Phi_2 + B\end{aligned}$$

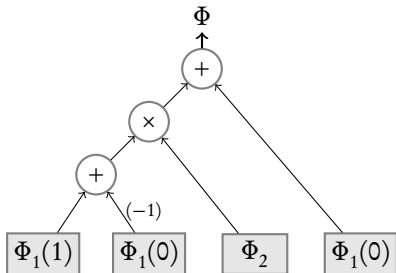
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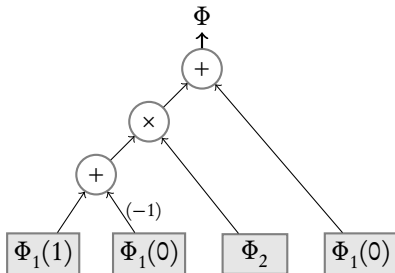
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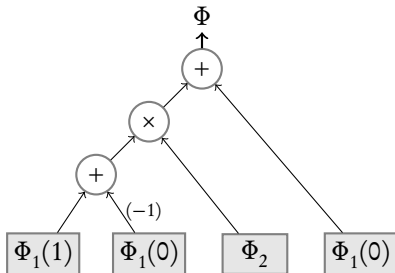
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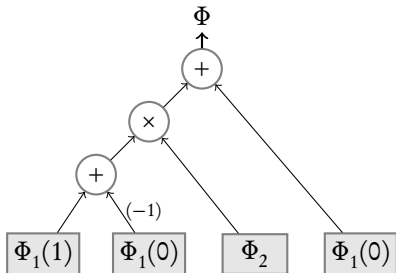
$$\begin{aligned} \text{Size}(s) &\leq 4 \cdot \text{Size}(2s/3) + O(1) \\ \text{Depth}(s) &\leq \text{Depth}(2s/3) + O(1) \end{aligned}$$

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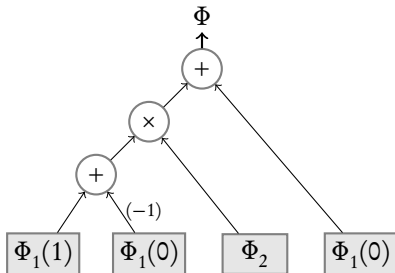
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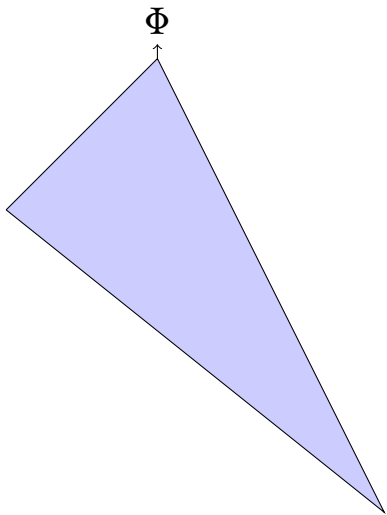
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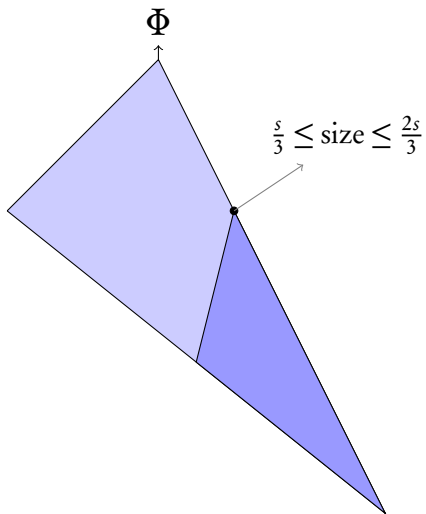
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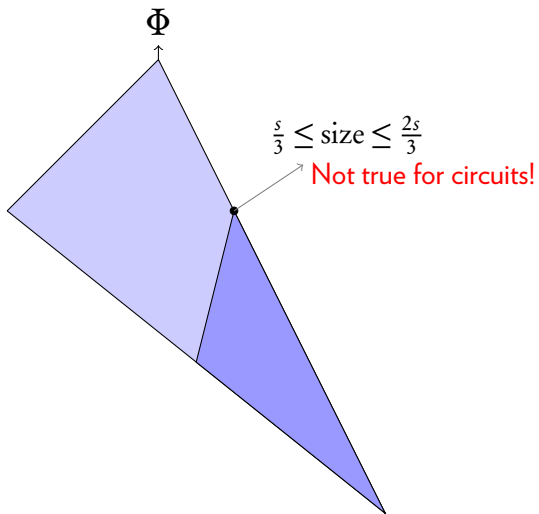
Adapting to circuits



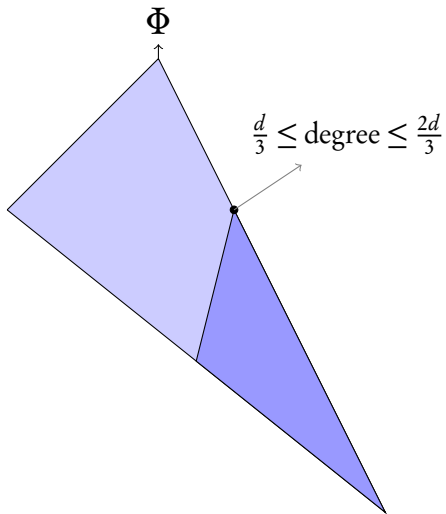
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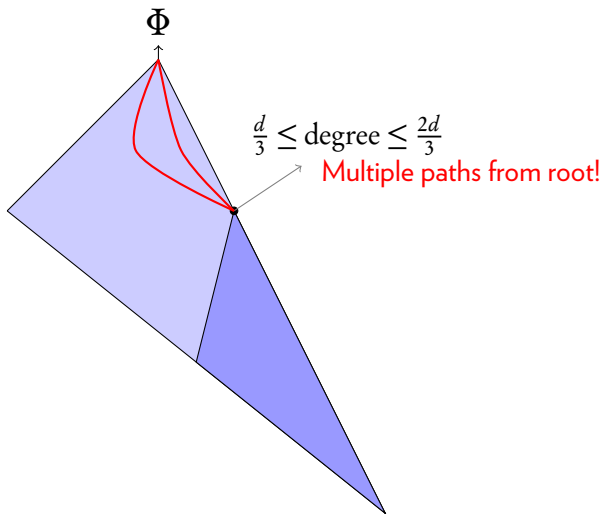
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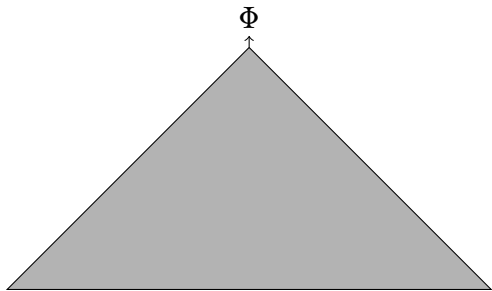
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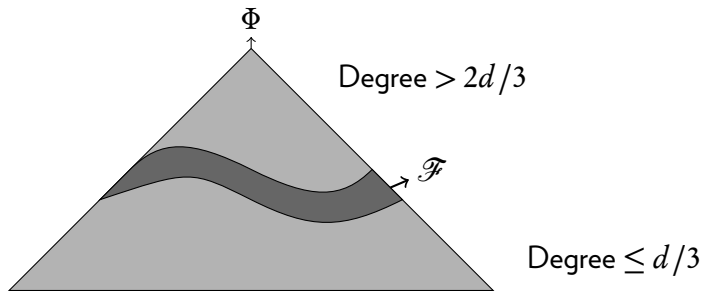
Adapting to circuits



Adapting to circuits: Attempt 1

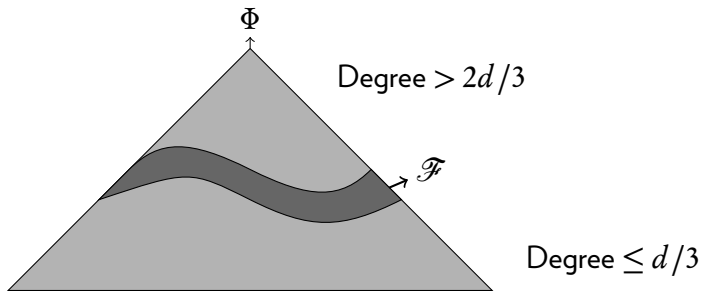


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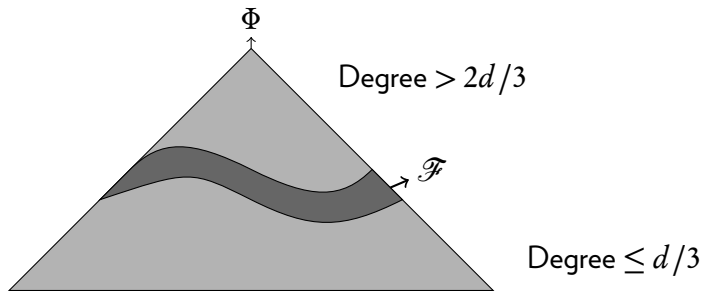
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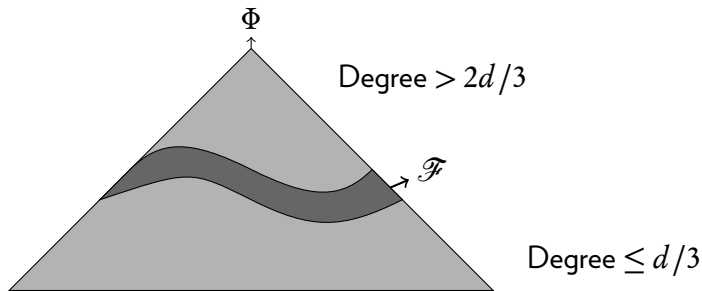


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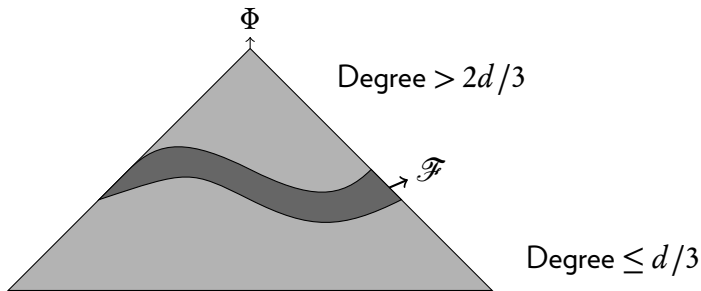


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Interpolate!

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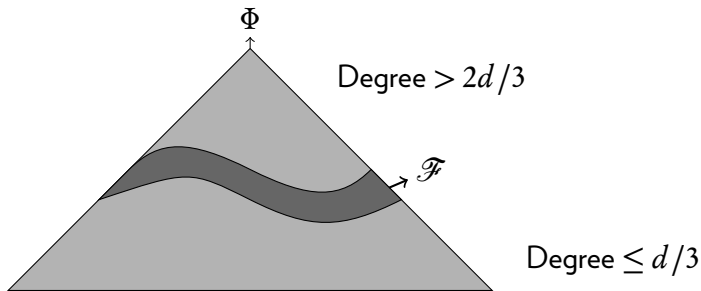


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$$\text{Depth}(d) = \text{Depth}(2d/3) + O(1)$$

Adapting to circuits: Attempt 1

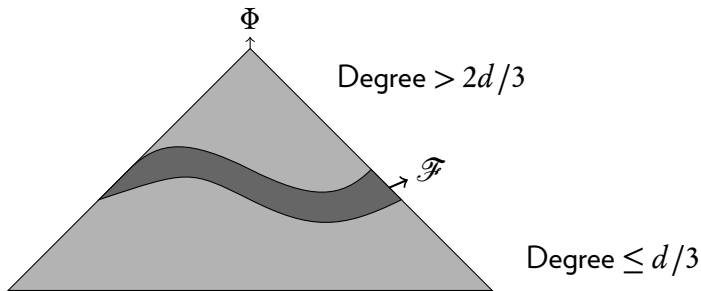


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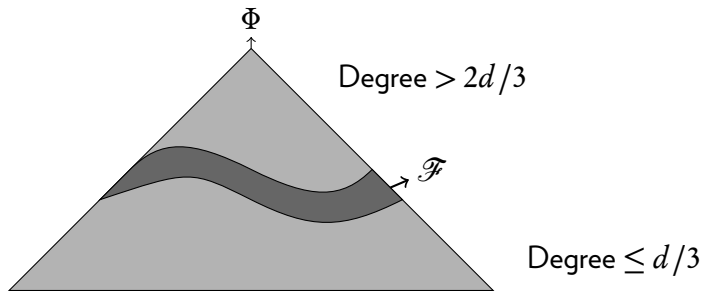
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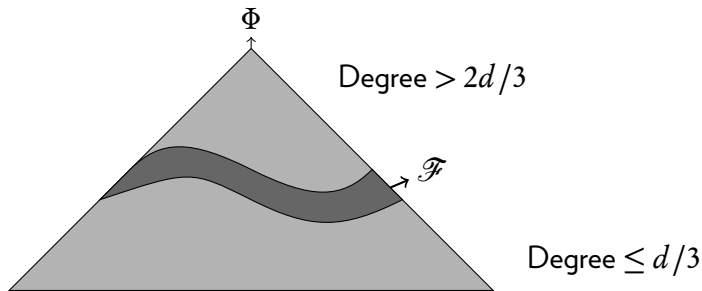
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Adapting to circuits: [Hyafil]



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$$[u : v] = \begin{cases} 0 & \text{if } u \text{ is a leaf} \\ 1 & \text{if } u = v \\ [u_1 : v] + [u_2 : v] & \text{if } u = u_1 + u_2 \\ [u_1] \cdot [u_2 : v] & \text{if } u_1 = u_1 \times u_2 \end{cases}$$

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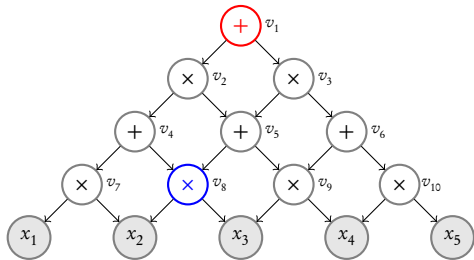
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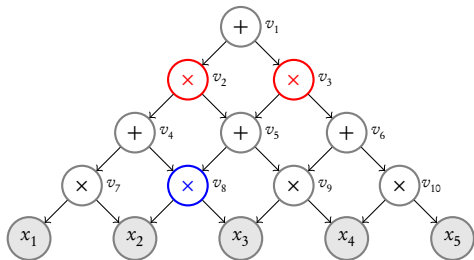
Works, but one needs to be a little careful with multiple paths. See [Shpilka-Yehudayoff]

An example



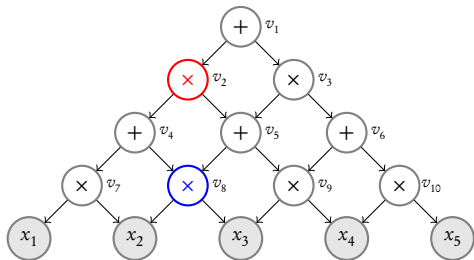
$$[v_1 : v_8] =$$

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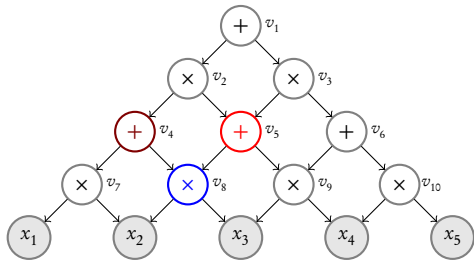
$$[v_1 : v_8] = [v_2 : v_8] + [v_3 : v_8]$$

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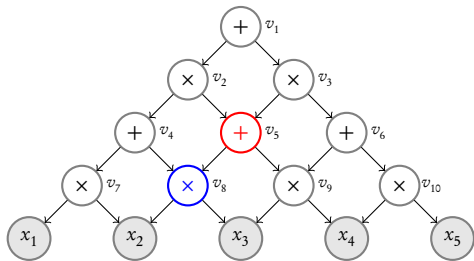
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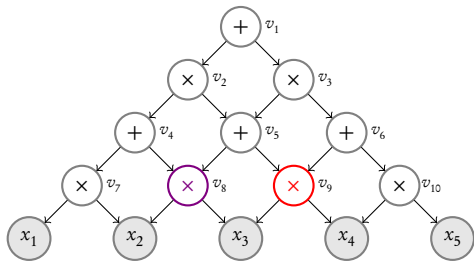
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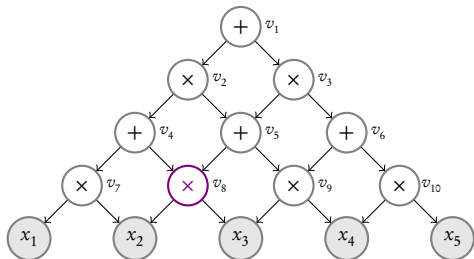
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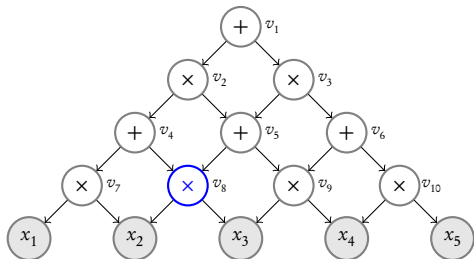
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[VSBR] *continued ...*

We want a set of nodes \mathcal{F} such that

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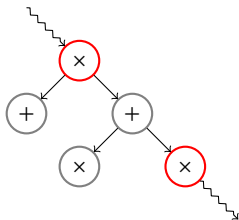
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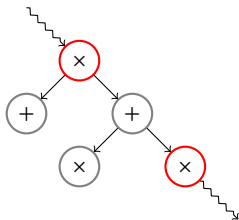


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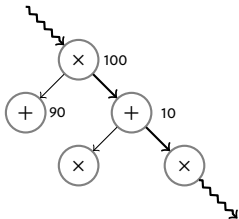
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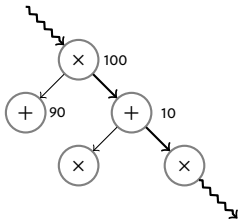
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Make the circuit *right heavy*.

[VSBR] *continued ...*

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Theorem ([Valiant-Skyum-Berkowitz-Rackoff])

If Φ is a size s circuit computing an n -variate degree d polynomial f , then there is a circuit Φ' computing f with the following properties.

- ▶ Every gate of Φ' computes some $[u : v]$,
- ▶ All addition gates have fan-in at most s^2 ,
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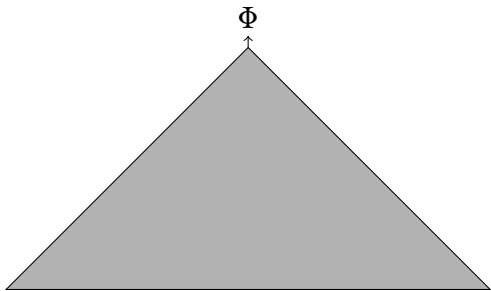
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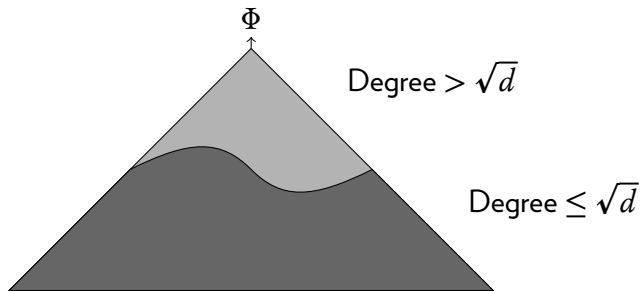
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Hence, the depth of Φ' is $O(\log d)$.

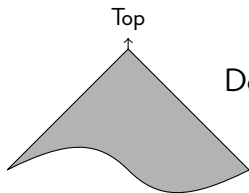
Reducing to depth four



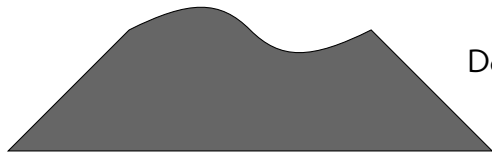
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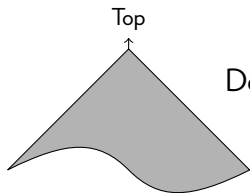


Degree $> \sqrt{d}$

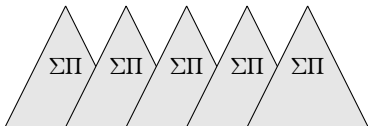


Degree $\leq \sqrt{d}$

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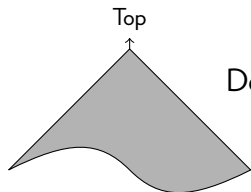
Degree $> \sqrt{d}$



Degree $\leq \sqrt{d}$

Size $\binom{n+\sqrt{d}}{\sqrt{d}}$ each

Reducing to depth four



Degree $> \sqrt{d}$



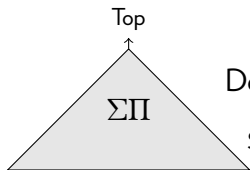
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Lemma ([Tavenas13])

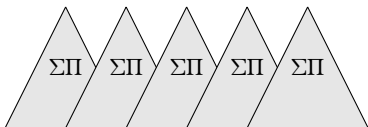
$$\deg(\text{Top}(z_1, \dots, z_s)) \leq 15\sqrt{d}$$

Reducing to depth four



Degree $> \sqrt{d}$

Size $\binom{s+15\sqrt{d}}{15\sqrt{d}}$



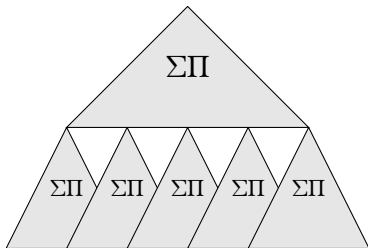
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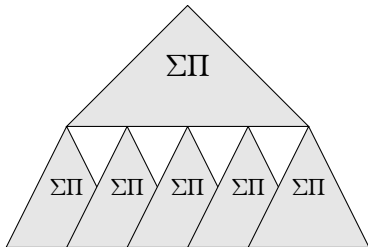


Theorem

Equivalent depth-4 circuit of size

$$s \binom{n + \sqrt{d}}{n} + \binom{s + 15\sqrt{d}}{s} = s^{O(\sqrt{d})}$$

Reducing to depth four



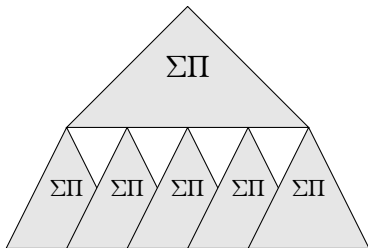
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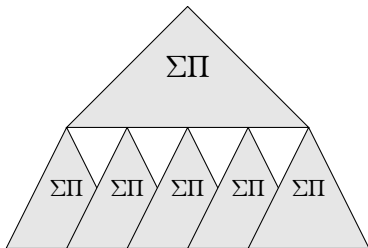
Theorem

Equivalent depth-4 circuit *with bottom fan-in at most \sqrt{d}* of size

$$s \binom{n + \sqrt{d}}{n} + \binom{s + 15\sqrt{d}}{s} = s^{O(\sqrt{d})}$$



Reducing to depth four



Theorem

Equivalent $\Sigma\Pi\Sigma\Pi^{[\sqrt{d}]}$ circuit of size

$$s \binom{n + \sqrt{d}}{n} + \binom{s + 15\sqrt{d}}{s} = s^{O(\sqrt{d})}$$

□

A different perspective

Let's start with [VSBR]

$$f = \sum_{i=1}^s f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5}$$

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How many iterations until all degrees are at most t ?

Number of iterations

$$g = \sum_{j=1}^s g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5}$$

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In each summand, at least two terms have degree at least $t/8$.

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How many factors of degree at least $t/8$?

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Number of iterations

$$g = \sum_{j=1}^s \underbrace{g_{j1}}_{\geq t/5} \cdot \underbrace{g_{j2}}_{\geq t/8} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5}$$

Observation

In each summand, at least two terms have degree at least $t/8$.

How many factors of degree at least $t/8$? At most $8d/t$.

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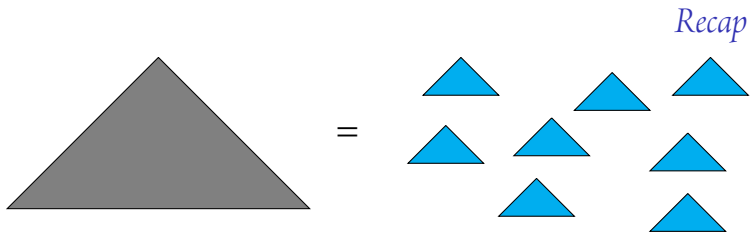
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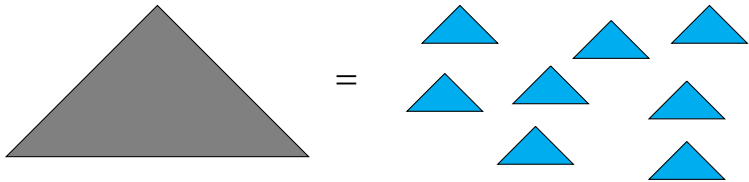
Final $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit has top fan-in at most $s^{O(d/t)}$. □




Theorem

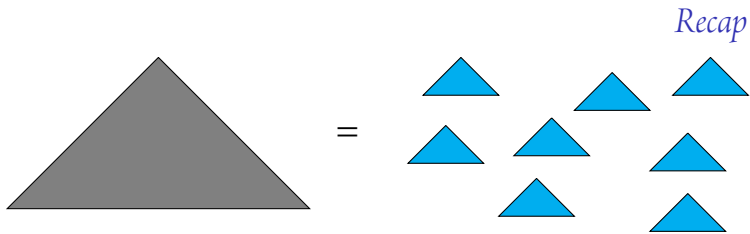
Every **small** circuit can be equivalently computed as a *sum* of **few** s

Recap



Theorem

Every **circuit of size s** can be equivalently computed as a *sum* of **few** s

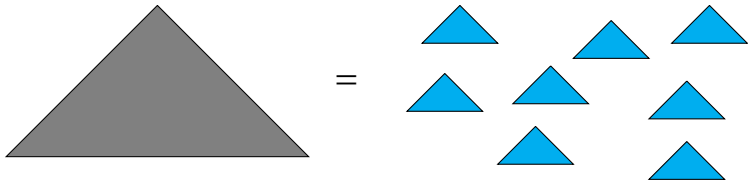


Theorem

Every circuit of size s can be equivalently computed as a sum of $s^{O(d/t)}$



Recap

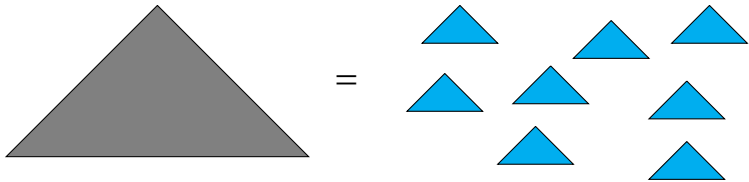


Theorem


Every **circuit of size s** can be equivalently computed as a *sum of $s^{O(d/t)}$*
▲ s , where

$$\triangle = \prod_{i=1}^{O(d/t)} Q_i \quad \deg(Q_i) \leq t$$

Recap




Theorem

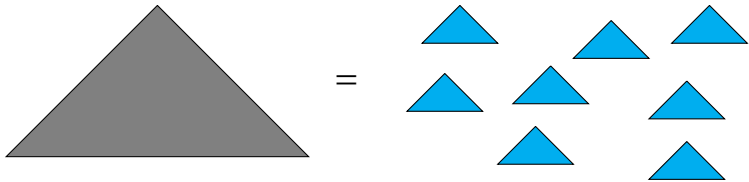
Every **circuit of size s** can be equivalently computed as a *sum* of $s^{O(d/t)}$ s, where

$$\text{blue triangle} = \prod_{i=1}^{O(d/t)} Q_i \quad \deg(Q_i) \leq t$$


Question

What are the s if we start with a **homogeneous formula of size s** ?

Recap



Theorem

Every **circuit of size s** can be equivalently computed as a *sum* of $s^{O(d/t)}$ , where

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What are the s if we start with a **depth 100 formula of size s** ?

A better starting point?

$$f = \sum_{i=1}^s f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5}$$

If we start with a homogeneous formula, can we do better?

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[Hrubes-Yehudayoff]: Yes!

Lemma ([Hrubes-Yehudayoff])

$$f = \sum_{i=1}^s f_{i1} \cdot f_{i2} \cdots f_{il} \quad \text{with} \quad \left(\frac{1}{3}\right)^j \cdot d < \deg(f_{ij}) \leq \left(\frac{2}{3}\right)^j \cdot d$$

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Proof

$$f = A \cdot \Phi_v + \Phi_{v=0}$$

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Proof

$$\begin{aligned} f &= A \cdot \Phi_v + \Phi_{v=0} \\ &= A \cdot \left(\sum_{i=1}^{s_1} g_{i1} \cdots g_{il} \right) + \left(\sum_{j=1}^{s_2} h_{j1} \cdots h_{jl} \right) \end{aligned}$$

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Reduction to depth four, again

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- ▶ The above expression is a $\Sigma\Pi\Sigma\Pi^{[2d/3]}$ circuit. We want a $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit.

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Reduction to depth four, again

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Proof: There are at least two terms of degree $t/9$. Yada Yada Yada \square

Reduction to depth four, again

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Yields a $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit of top fan-in $s^{O(d/t)}$.

Wait... what's different?

For circuits:

$$f = \sum_{i=1}^s f_{i1} \cdot f_{i2} \cdots f_{i5}$$

For homogeneous formulas:

$$f = \sum_{i=1}^s f_{i1} \cdot f_{i2} \cdots f_{il}$$

Wait... what's different?

For circuits:

$$f = \sum_{i=1}^s \left(\sum_{j=1}^s g_{i1} \cdots g_{i5} \right) \cdot f_{i2} \cdots f_{i5}$$

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$$f = \sum_{i=1}^{s^2} f_{i1} \cdot f_{i2} \cdots f_{i9}$$

For homogeneous formulas:

$$f = \sum_{i=1}^{s^2} f_{i1} \cdot f_{i2} \cdots f_{i(2\ell)}$$

Wait... what's different?

For circuits:

$$f = \sum_{i=1}^{s^4} f_{i1} \cdot f_{i2} \cdots f_{i13}$$

For homogeneous formulas:

$$f = \sum_{i=1}^{s^4} f_{i1} \cdot f_{i2} \cdots f_{i(3\ell)}$$

Wait... what's different?

For circuits:

$$f = \sum_{i=1}^{s^r} f_{i1} \cdot f_{i2} \cdots f_{i(4r+1)}$$

For homogeneous formulas:

$$f = \sum_{i=1}^{s^r} f_{i1} \cdot f_{i2} \cdots f_{i(rl)}$$

Wait... what's different?

For circuits:

$$f = \sum_{i=1}^{s^r} f_{i_1} \cdot f_{i_2} \cdots f_{i_{(4r+1)}}$$

a $\Sigma\Pi^{[O(d/t)]}\Sigma\Pi^{[t]}$ circuit

For homogeneous formulas:

$$f = \sum_{i=1}^{s^r} f_{i_1} \cdot f_{i_2} \cdots f_{i_{(r\ell)}}$$

a $\Sigma\Pi^{[O(d/t)\cdot\log d]}\Sigma\Pi^{[t]}$ circuit

Wait... what's different?

For circuits:


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a $\Sigma\Pi^{[O(d/t) \cdot \log d]}\Sigma\Pi^{[t]}$ circuit

Here, s factorize more


Is that a big deal?

Circuit class


No. factors of 

Lower bound


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Circuit class	No. factors of 	Lower bound
Hom. $\Sigma\Pi\Sigma$	$O(n)$	$n^{\Omega(n)}$


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$\Sigma\Pi\Sigma\Pi^{[t]}$	$O(d/t)$	$n^{\Omega(d/t)}$

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$\Sigma\Pi\Sigma\Pi^{[t]}$	$O(d/t)$	$n^{\Omega(d/t)}$
Hom. $\Sigma\Pi\Sigma\Pi$	$O(\sqrt{d})$	$n^{O(\sqrt{d})}$

Is that a big deal?

Circuit class	No. factors of 	Lower bound
Hom. $\Sigma\Pi\Sigma$	$O(n)$	$n^{\Omega(n)}$
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$\Sigma\Pi\Sigma\Pi^{[t]}$	$O(d/t)$	$n^{\Omega(d/t)}$
Hom. $\Sigma\Pi\Sigma\Pi$	$O(\sqrt{d})$	$n^{O(\sqrt{d})}$

Wishful Question: Can we get an $n^{\Omega(\log n)}$ lower bound for homogeneous formulas, using current techniques (with slight modifications)?

Reduction to Depth-3 Circuits

▶▶ No time!

Road map

$\Sigma^{\sqrt{d}} \Pi \Sigma \Pi^{\sqrt{d}}$
circuits



$\Sigma^{\sqrt{d}} \wedge \Sigma \wedge \Sigma^{\sqrt{d}}$
circuits



$\Sigma \Pi \Sigma$
circuits

Road map

$\Sigma^{\sqrt{d}} \Pi^{\sqrt{d}} \Sigma \Pi$
circuits



App. of Ryser's formula

$\Sigma^{\sqrt{d}} \wedge \Sigma^{\sqrt{d}} \wedge \Sigma$
circuits



$\Sigma \Pi \Sigma$
circuits

Road map

$$\sum^{\sqrt{d}} \Pi \sum^{\sqrt{d}} \Pi$$

circuits



App. of Ryser's formula

$$\sum^{\sqrt{d}} \wedge \sum^{\sqrt{d}} \wedge \sum$$

circuits



[Saxena]'s duality trick

$$\sum \Pi \sum$$

circuits

Road map

$$\sum^{\sqrt{d}} \Pi \sum^{\sqrt{d}} \Pi$$

circuits

Only over \mathbb{Q}, \mathbb{R} etc.

App. of Ryser's formula

$$\sum^{\sqrt{d}} \wedge \sum^{\sqrt{d}} \wedge \sum$$

circuits

Heavily non-homogeneous

[Saxena]'s duality trick

$$\sum \Pi \sum$$

circuits

Step 1: ΣΠΣΠ to ΣΛΣΛΣ

Step 1: $\Sigma\Pi\Sigma\Pi$ to $\Sigma\Lambda\Sigma\Lambda\Sigma$

Recall Ryser's formula:

$$\text{Perm}_n \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^n \sum_{j \in S} x_{ij}$$

Step 1: $\Sigma\Pi\Sigma\Pi$ to $\Sigma\Lambda\Sigma\Lambda\Sigma$

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$$\text{Perm}_n \begin{bmatrix} x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ x_1 & \cdots & x_n \end{bmatrix} = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^n \sum_{j \in S} x_j$$

Step 1: $\Sigma\Pi\Sigma\Pi$ to $\Sigma\Lambda\Sigma\Lambda\Sigma$

Recall Ryser's formula:

$$\text{Perm}_n \begin{bmatrix} x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ x_1 & \cdots & x_n \end{bmatrix} = \sum_{S \subseteq [n]} (-1)^{n-|S|} \left(\sum_{j \in S} x_j \right)^n$$

Step 1: $\Sigma\Pi\Sigma\Pi$ to $\Sigma\wedge\Sigma\wedge\Sigma$

Recall Ryser's formula:

$$n! \cdot x_1 \dots x_n = \sum_{S \subseteq [n]} (-1)^{n-|S|} \left(\sum_{j \in S} x_j \right)^n$$

Step 1: $\Sigma\Pi\Sigma\Pi$ to $\Sigma\wedge\Sigma\wedge\Sigma$

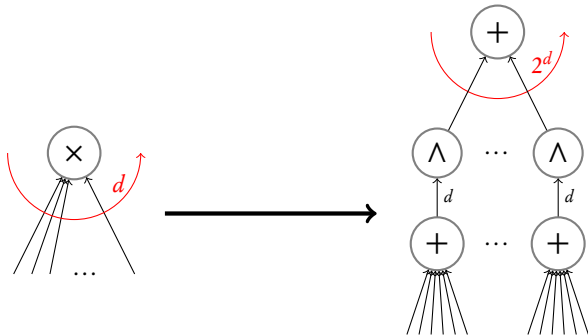
[Fischer]:

$$n! \cdot x_1 \dots x_n = \sum_{S \subseteq [n]} (-1)^{n-|S|} \left(\sum_{j \in S} x_j \right)^n$$

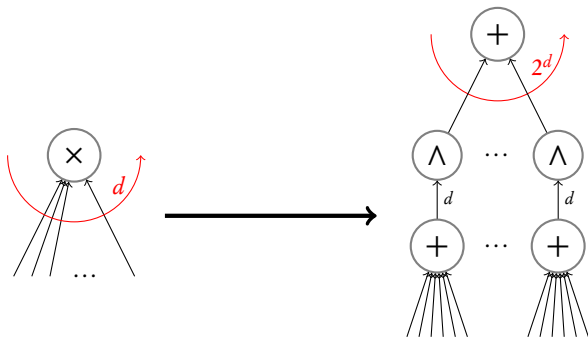
Step 1: $\Sigma\Pi\Sigma\Pi$ to $\Sigma\Lambda\Sigma\Lambda\Sigma$

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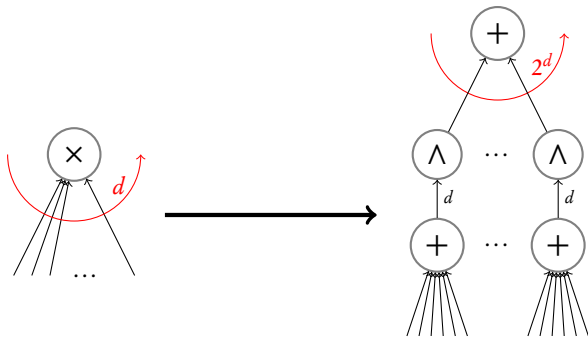


Step 1: $\Sigma\Pi\Sigma\Pi$ to $\Sigma\Lambda\Sigma\Lambda\Sigma$



$$\overset{d}{\prod} \longrightarrow \overset{2^d}{\Sigma} \overset{d}{\wedge} \overset{d}{\Sigma}$$

Step 1: $\Sigma\Pi\Sigma\Pi$ to $\Sigma\wedge\Sigma\wedge\Sigma$



$$\overset{d}{\Pi} \rightarrow \overset{2^d}{\Sigma} \overset{d}{\wedge} \overset{d}{\Sigma}$$

$$\overset{\sqrt{d}}{\Sigma} \overset{\sqrt{d}}{\Pi} \overset{\sqrt{d}}{\Sigma} \overset{\sqrt{d}}{\Pi} \text{ of size } s \rightarrow \overset{\sqrt{d}}{\Sigma} \overset{\sqrt{d}}{\wedge} \overset{\sqrt{d}}{\Sigma} \overset{\sqrt{d}}{\wedge} \overset{\sqrt{d}}{\Sigma} \text{ of size } 2^{O(\sqrt{d})} \cdot s$$

Road map

$\overset{\sqrt{d}}{\Sigma} \overset{\sqrt{d}}{\Pi} \Sigma \Pi$
circuits



$\overset{\sqrt{d}}{\Sigma} \overset{\sqrt{d}}{\wedge} \Sigma \overset{\sqrt{d}}{\wedge} \Sigma$
circuits



$\Sigma \Pi \Sigma$
circuits

Road map

$\Sigma^{\sqrt{d}} \Pi \Sigma^{\sqrt{d}} \Pi$
circuits



$\Sigma^{\sqrt{d}} \wedge \Sigma^{\sqrt{d}} \wedge \Sigma$
circuits

$\Sigma \Pi \Sigma$
circuits

Step 2: $\Sigma \Lambda \Sigma \Lambda \Sigma$ to $\Sigma \Pi \Sigma$

$$\Sigma \overset{\sqrt{d}}{\wedge} \Sigma \overset{\sqrt{d}}{\wedge} \Sigma$$

Step 2: $\Sigma \Lambda \Sigma \Lambda \Sigma$ to $\Sigma \Pi \Sigma$

$$\Sigma \overset{\sqrt{d}}{\wedge} \Sigma \overset{\sqrt{d}}{\wedge} \Sigma$$

ℓ

Step 2: $\Sigma \Lambda \Sigma \Lambda \Sigma$ to $\Sigma \Pi \Sigma$

$$\Sigma \overset{\sqrt{d}}{\wedge} \Sigma \overset{\sqrt{d}}{\wedge} \Sigma$$

$$\ell^{\sqrt{d}}$$

Step 2: $\Sigma \wedge \Sigma \wedge \Sigma$ to $\Sigma \Pi \Sigma$

$$\Sigma \wedge^{\sqrt{d}} \Sigma \wedge^{\sqrt{d}} \Sigma$$

$$\ell_1^{\sqrt{d}} + \dots + \ell_s^{\sqrt{d}}$$

Step 2: $\Sigma \wedge \Sigma \wedge \Sigma$ to $\Sigma \Pi \Sigma$

$$\Sigma \wedge^{\sqrt{d}} \Sigma \wedge^{\sqrt{d}} \Sigma$$

$$\left(\ell_1^{\sqrt{d}} + \dots + \ell_s^{\sqrt{d}} \right)^{\sqrt{d}}$$

Step 2: $\Sigma \wedge \Sigma \wedge \Sigma$ to $\Sigma \Pi \Sigma$

$$\Sigma \wedge^{\sqrt{d}} \Sigma \wedge^{\sqrt{d}} \Sigma$$

$$\sum_i \left(\ell_{i1}^{\sqrt{d}} + \dots + \ell_{is}^{\sqrt{d}} \right)^{\sqrt{d}}$$

Step 2: $\Sigma \wedge \Sigma \wedge \Sigma$ to $\Sigma \Pi \Sigma$

$$C = \sum_i \left(\ell_{i1}^{\sqrt{d}} + \dots + \ell_{is}^{\sqrt{d}} \right)^{\sqrt{d}}$$

Step 2: $\Sigma\Lambda\Sigma\Lambda\Sigma$ to $\Sigma\Pi\Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \dots + \ell_s^{\sqrt{d}} \right)^{\sqrt{d}}$$

Step 2: $\Sigma \wedge \Sigma \wedge \Sigma$ to $\Sigma \Pi \Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \dots + \ell_s^{\sqrt{d}} \right)^{\sqrt{d}}$$

Lemma ([Saxena])

There exists univariate polynomials f_{ij} 's of degree at most d such that

$$(x_1 + \dots + x_s)^d = \sum_{i=1}^{sd+1} \prod_{j=1}^s f_{ij}(x_j)$$

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Sketch of a proof by Gupta-Forbes-Shpilka

$$P_{\mathbf{x}}(t) = (1 + x_1 t) \dots (1 + x_s t) = 1 + \ell t + (\text{higher degree terms})$$

Step 2: $\Sigma \wedge \Sigma \wedge \Sigma$ to $\Sigma \Pi \Sigma$

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Sketch of a proof by Gupta-Forbes-Shpilka

$$P_{\mathbf{x}}(t) - 1 = \ell t + (\text{higher degree terms})$$

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Sketch of a proof by Gupta-Forbes-Shpilka

$$(P_{\mathbf{x}}(t) - 1)^d = \ell^d t^d + \text{(higher degree terms)}$$

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Sketch of a proof by Gupta-Forbes-Shpilka

$$(P_{\mathbf{x}}(t) - 1)^d = \ell^d t^d + \text{(higher degree terms)}$$

Interpolate!

$(P_{\mathbf{x}}(t) - 1)^d$ expanded is a sum of $(d + 1)$ product of univariates. □

Step 2: $\Sigma \wedge \Sigma \wedge \Sigma$ to $\Sigma \Pi \Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \dots + \ell_s^{\sqrt{d}} \right)^{\sqrt{d}}$$

$$(x_1 + \dots + x_s)^{\sqrt{d}} = \sum_i^{\text{poly}(s,d)} \prod_{j=1}^s f_{ij}(x_j)$$

Step 2: $\Sigma \wedge \Sigma \wedge \Sigma$ to $\Sigma \Pi \Sigma$

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where $\tilde{f}_{ij}(t) := f_{ij}(t^{\sqrt{d}})$

Step 2: $\Sigma \wedge \Sigma \wedge \Sigma$ to $\Sigma \Pi \Sigma$

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Note that $\tilde{f}_{ij}(t)$ is a **univariate** polynomial

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$$\tilde{f}_{ij}(t) = \prod_{k=1}^d (t - \zeta_{ijk})$$

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... a $\Sigma\Pi\Sigma$ circuit of $\text{poly}(s, d)$ size.

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ to $\Sigma\Pi\Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \dots + \ell_s^{\sqrt{d}} \right)^{\sqrt{d}}$$

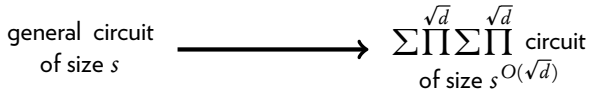
$$\begin{aligned} \left(\ell_1^{\sqrt{d}} + \dots + \ell_s^{\sqrt{d}} \right)^{\sqrt{d}} &= \sum_i^{\text{poly}(s,d)} \prod_{j=1}^s f_{ij} \left(\ell_j^{\sqrt{d}} \right) \\ &= \sum_i^{\text{poly}(s,d)} \prod_{j=1}^s \tilde{f}_{ij}(\ell_j) \\ &= \sum_i^{\text{poly}(s,d)} \prod_{j=1}^s \prod_{k=1}^d \left(\ell_j - \zeta_{ijk} \right) \end{aligned}$$

... a $\Sigma\Pi\Sigma$ circuit of $\text{poly}(s, d)$ size **and degree sd** .

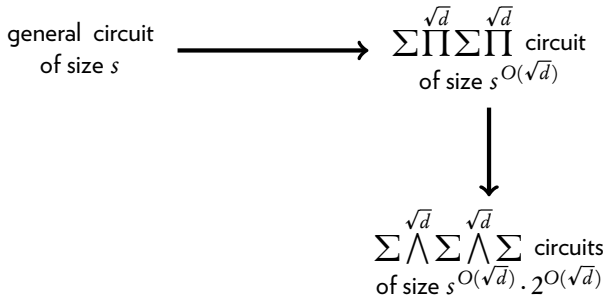
Putting it together

general circuit
of size s

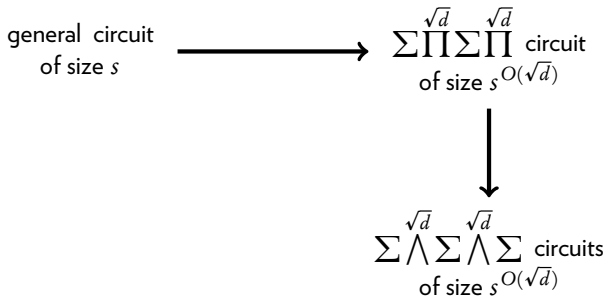
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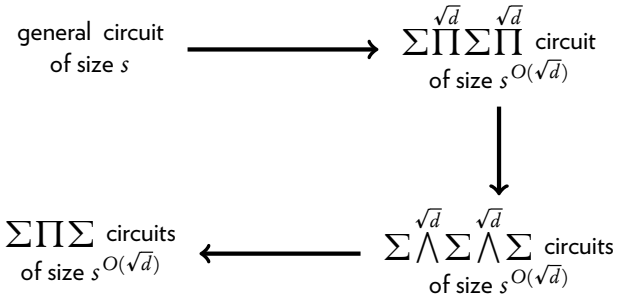
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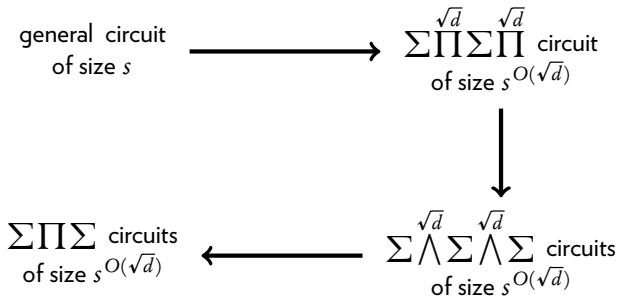
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Putting it together

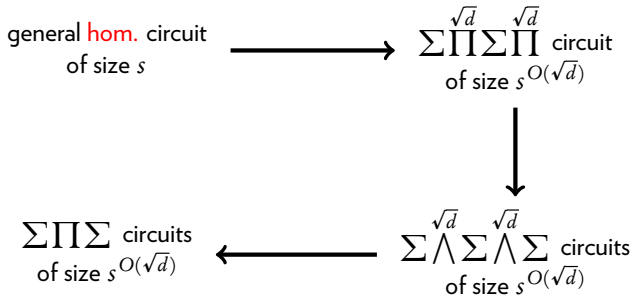


Putting it together



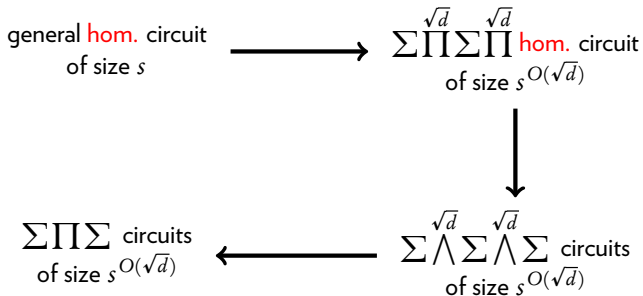
Question: Where should one try to prove lower bounds?

Putting it together



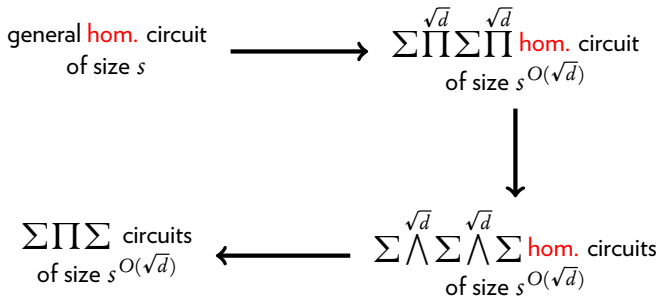
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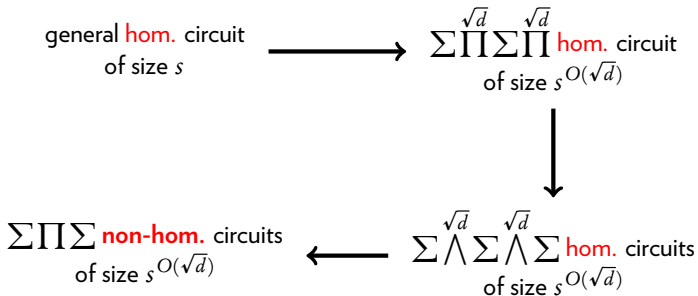
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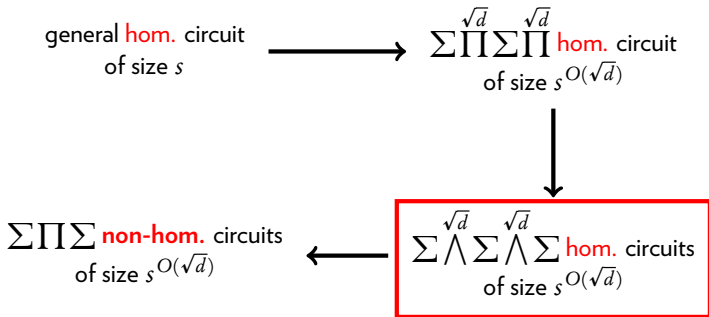
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Summary

- ▶ Depth reduction can manifest in many forms. Finding the right building block is sometimes crucial.
- ▶ A *slightly* different proof of [Tavenas] yields a possible useful building block for homogeneous formulas with more factors.
- ▶ *Maybe* we can get $n^{\Omega(\log n)}$ lower bounds via modified shifted-partials.
- ▶ Can we say something similar about $\Sigma\Pi\Sigma\Pi^{[t]}$ circuits obtained from ABPs?

Call for contributors

A git survey on arithmetic circuit lower bounds:

<https://github.com/dasarpmar/lowerbounds-survey/>

Dankeschön