Approaching the chasm at depth four

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CCC 2013

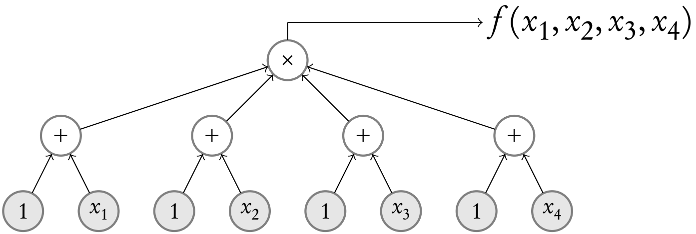
June 2013, Palo Alto

$$\begin{aligned} f(x_1,x_2,x_3,x_4) \quad &= \quad 1+x_1+x_2+x_3+x_4 \\ &\quad +x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4 \\ &\quad +x_2x_3x_4+x_1x_3x_4+x_1x_2x_4+x_1x_2x_3 \\ &\quad +x_1x_2x_3x_4 \end{aligned}$$

$$\begin{aligned} f(x_1, x_2, x_3, x_4) &= 1 + x_1 + x_2 + x_3 + x_4 \\ &\quad + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\ &\quad + x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 + x_1x_2x_3 \\ &\quad + x_1x_2x_3x_4 \\ &= (1 + x_1)(1 + x_2)(1 + x_3)(1 + x_4) \end{aligned}$$

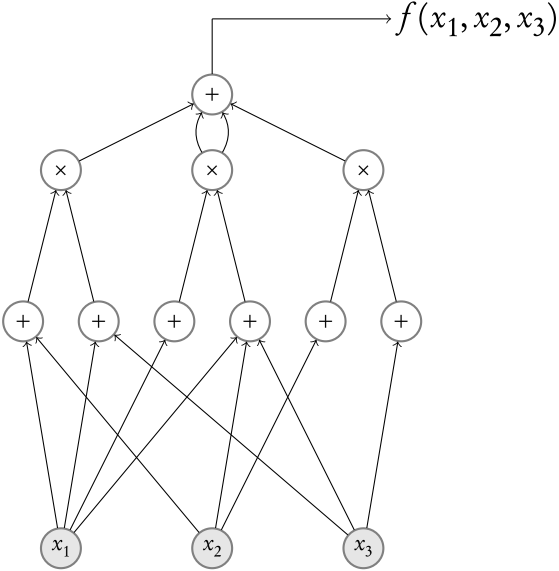
... certainly a more compact representation.

Arithmetic Formulae



- Tree
- Leaves containing variables or constants

Arithmetic Circuits



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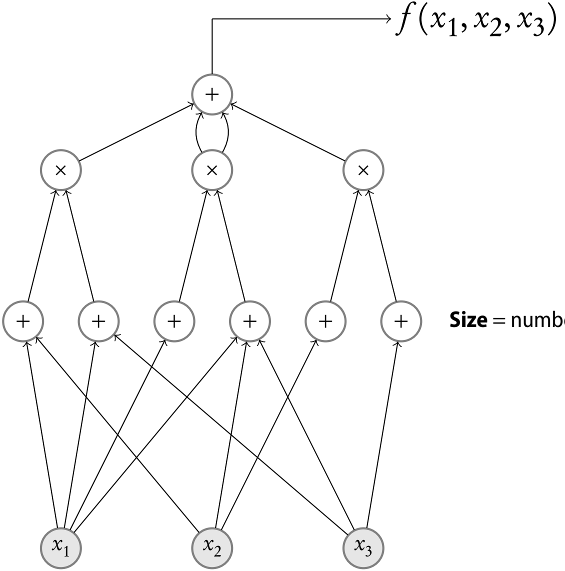
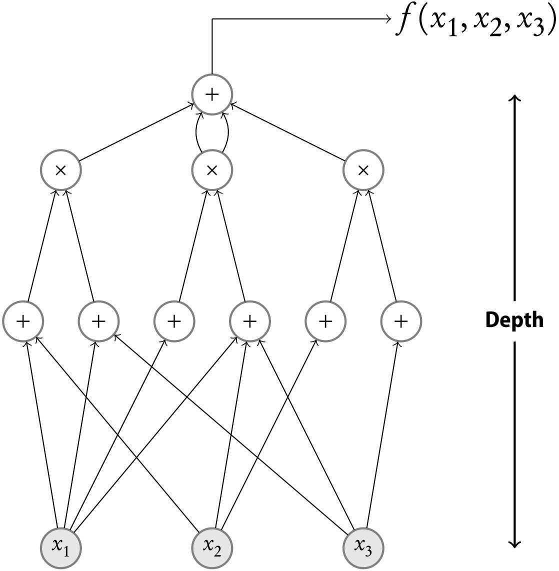


Size = number of gates

Arithmetic Circuits



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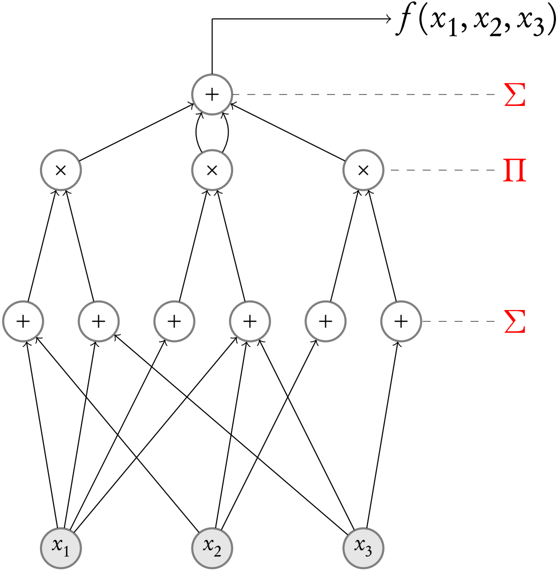

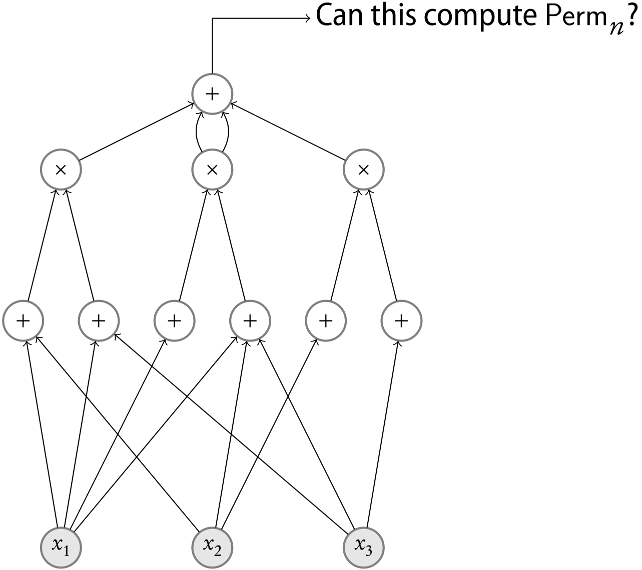
Can this compute Perm_n?
+
×
×
×
+
+
+
+
+
+
x_1
x_2
x_3

The illustrious siblings

$$\mathsf{Det}_n(x_{11},\ldots,x_{nn}) \;=\; \sum_{\sigma\in S_n} \mathsf{sign}(\sigma)\cdot x_{1\sigma(1)}\ldots x_{n\sigma(n)}$$

$$\mathsf{Perm}_n(x_{11},\ldots,x_{nn}) \;=\; \sum_{\sigma\in S_n} x_{1\sigma(1)}\ldots x_{n\sigma(n)}$$

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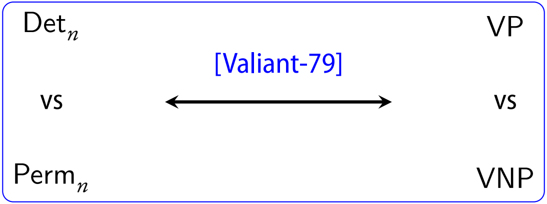

Known lower bounds

Model	**Lower bound**	
General circuits	$\Omega(n \log n)$	[Baur-Strassen-83]
General formulas	$\Omega(n^3)$	[Kalorkoti-85]

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General circuits	$\Omega(n \log n)$	[Baur-Strassen-83]
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Depth-2 circuits	$2^{\Omega(n \log n)}$	(Trivial!)
Homogeneous Depth-3 circuits	$2^{\Omega(n)}$	[Nisan-Wigderson-97]
Depth-3 circuits over finite fields	$2^{\Omega(n)}$	[Grigoriev-Karpinski-98]
Depth-3 circuits	$\Omega(n^2)$	[Shpilka-Wigderson-01]

Known lower bounds

Model	Lower bound	
Multilinear formula	$n^{\Omega(\log n)}$	[Raz-09]
Constant depth, *multilinear* formula	$2^{\tilde{\Omega}(n^{1/d})}$	[Raz-Yehudayoff-09]
Monotone Circuits	$2^{\Omega(n)}$	[Jerrum-Snir-82]

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Summary: No lower bounds known beyond depth-3, unless other restrictions are imposed.

Theorem ([Agrawal-Vinay-08, Koiran-12, Sébastien-13])

Can be computed by		*Can be computed by*
arithmetic circuits	$\longrightarrow$	*depth-4 circuits*
of "small" size		*of "not-too-large" size*

Chasm at depth-4

Theorem ([Agrawal-Vinay-08, Koiran-12, Sébastien-13])

Can be computed by		*Can be computed by*
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of $\mathrm{poly}(n, d)$ *size*		*of* $n^{O(\sqrt{d})}$ *size*

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Can be computed by arithmetic circuits of poly(n, d) *size*	$\longrightarrow$	*Can be computed by depth-4 circuits of* $n^{O(\sqrt{d})}$ *size*

(Or)

Cannot be computed by arithmetic circuits of poly(n, d) *size*	$\longleftarrow$	*Cannot be computed by depth-4 circuits of* $n^{O(\sqrt{d})}$ *size*

Chasm at depth-3

Theorem ([Gupta-Kamath-Kayal-S-13])

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of $\text{poly}(n, d)$ size		of $n^{O(\sqrt{d})}$ size

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Theorem ([Agrawal-Vinay-08, Koiran-12, Sébastien-13])

Perm_n *cannot be computed by* $\Sigma\Pi\Sigma\Pi$ *circuits* *of* $n^{O(\sqrt{n})}$ *size*

$\longrightarrow$

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Chasm at depth-4

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$$C \quad = \quad \sum_{i=1}^{s} Q_{i1} \dots Q_{i\sqrt{n}} \quad \text{and } \deg(Q_{ij}) = \sqrt{n}.$$

Chasm at depth-4

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Theorem ([Gupta-Kamath-Kayal-S])

Perm_n *(or* Det_n*) requires*

$\Sigma\Pi^{[\sqrt{n}]}\Sigma\Pi^{[\sqrt{n}]}$ *circuits*

of $2^{\Omega(\sqrt{n})}$ *size*

Elevator at depth-4

Theorem ([Agrawal-Vinay-08, Koiran-12, Sébastien-13])

Perm_n *cannot be computed by* $\Sigma\Pi^{[\sqrt{n}]}\Sigma\Pi^{[\sqrt{n}]}$ *circuits of* $n^{O(\sqrt{n})}$ *size* $\longrightarrow$ Perm_n *cannot be computed by arithmetic circuits of* $\text{poly}(n)$ *size*

Theorem ([Gupta-Kamath-Kayal-S])

Perm_n *(or* Det_n*) requires* $\Sigma\Pi^{[\sqrt{n}]}\Sigma\Pi^{[\sqrt{n}]}$ *circuits of* $2^{\Omega(\sqrt{n})}$ *size*

"Natural" lower bound proofs

Construct a map $\Gamma : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{N}$, that assigns a number to every polynomial such that:

1. If f is computable by "small" circuits, then $\Gamma(f)$ is "small".
2. For the desired polynomial f we wish to show a lower bound, then $\Gamma(f)$ is "large".

“Natural” lower bound proofs

Construct a map $\Gamma : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{N}$, that assigns a number to every polynomial such that:

Usually sub-additive: $\Gamma(f_1 + f_2) \quad \leq \quad \Gamma(f_1) + \Gamma(f_2)$

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   Typically done by showing f can be written as a "small" sum *building blocks*, and that each *building block* have small Γ.
2. For the desired polynomial f we wish to show a lower bound, then $\Gamma(f)$ is "large".
   Hence, should require **lots** of building blocks to compute it.

Weakness of building blocks

$$C \quad = \quad Q_{11} \ldots Q_{1d} + \ldots + Q_{s1} \ldots Q_{sd}$$

Weakness of building blocks

$$T = Q_1 \dots Q_d$$

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$$\partial^{=k}(Q_1 \ldots Q_d) = \begin{array}{c} (\partial Q_1) \ldots (\partial Q_k) Q_{k+1} \ldots Q_d \\ + \\ \vdots \\ + \\ Q_1 \ldots Q_{d-k} (\partial Q_{d-k+1}) \ldots (\partial Q_d) \end{array}$$

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Large gcd

Weakness of building blocks

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$$(\partial Q_1) \ldots (\partial Q_k) Q_{k+1} \ldots Q_d \cdot A$$

Many solutions to: $+$ $= 0$

$$Q_1 \ldots Q_{d-k} (\partial Q_{d-k+1}) \ldots (\partial Q_d) \cdot B$$

The Complexity Measure

$$\mathbf{x}^{\leq \ell} \stackrel{\text{def}}{=} \text{Set of monomials of degree bounded by } \ell$$

$$\partial^{=k}(f) \stackrel{\text{def}}{=} \text{Set of } k\text{-th order partial derivatives of } f$$

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$$\Gamma_{k,\ell}(f) \stackrel{\text{def}}{=} \dim\left\{\mathbf{x}^{\leq \ell} \cdot \partial^{=k}(f)\right\}$$

dimension of low-degree combinations of $\partial^{=k}(f)$

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Fact from Algebraic Geometry

Growth of $\Gamma_{k,\ell}(f)$ describes the geometry of roots of f with multiplicity k. (Hilbert polynomial)

Upper bound for $\Sigma\Pi^{[\sqrt{n}]}\Sigma\Pi^{[\sqrt{n}]}$ *circuits*

$$\begin{aligned} C &= \sum_{i=1}^{s} Q_{i1}\ldots Q_{i\sqrt{n}} \qquad \deg(Q_{ij}) = \sqrt{n} \\ \Gamma_{k,\ell}(C) &= \dim\left\{\mathbf{x}^{\leq \ell}\partial^{=k}(C)\right\} \end{aligned}$$

Upper bound for $\Sigma\Pi^{[\sqrt{n}]}\Sigma\Pi^{[\sqrt{n}]}$ *circuits*

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\Gamma_{k,\ell}(C) &= \dim\left\{\mathbf{x}^{\le \ell}\partial^{=k}(C)\right\} \\
&\le s \cdot \dim\left\{\mathbf{x}^{\le \ell}\partial^{=k}(Q_1 \ldots Q_{\sqrt{n}})\right\}
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\end{aligned}
$$

$$
\partial^{=k}(T) \in \mathbb{F}\text{-span}\left\{\left(\prod_{i\notin S} Q_i\right)\partial^{=k}\left(\prod_{i\in S} Q_i\right) : \begin{array}{c} S\subseteq[\sqrt{n}] \\ |S| = k \end{array}\right\}
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Degree at most $(\sqrt{n}k - k)$

$$\partial^{=k}(T) \in \mathbb{F}\text{-span}\left\{ \left(\prod_{i \notin S} Q_i\right) \partial^{=k} \left(\prod_{i \in S} Q_i\right) \;:\; \begin{array}{c} S \subseteq [\sqrt{n}] \\ |S| = k \end{array} \right\}$$

Upper bound for $\Sigma\Pi^{[\sqrt{n}]}\Sigma\Pi^{[\sqrt{n}]}$ circuits

$$C = \sum_{i=1}^{s} Q_{i1}\ldots Q_{i\sqrt{n}} \qquad \deg(Q_{ij}) = \sqrt{n}$$

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$$\leq s\cdot\dim\left\{\mathbf{x}^{\leq \ell}\partial^{=k}(Q_1\ldots Q_{\sqrt{n}})\right\}$$

$$\partial^{=k}(T) \in \mathbb{F}\text{-span}\left\{\left(\prod_{i\notin S} Q_i\right)\cdot \mathbf{x}^{\leq(\sqrt{n}k-k)} : \begin{array}{c} S\subseteq[\sqrt{n}] \\ |S| = k \end{array}\right\}$$

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&\leq s\cdot \boxed{\text{Expression 1}}
\end{aligned}
$$

Lower bound for $\Gamma_{k,\ell}(\mathsf{Perm}_n)$

$$
\begin{aligned}
\Gamma_{k,\ell}(\mathsf{Perm}_n) &= \dim\left\{\mathbf{x}^{\leq \ell}\,\partial^{=k}\mathsf{Perm}_n\right\} \\
&\geq \#\left\{\begin{array}{l}\text{distinct leading monomials} \\ \text{in} \quad \mathbf{x}^{\leq \ell}\cdot\partial^{=k}(\mathsf{Perm}_n)\end{array}\right\}
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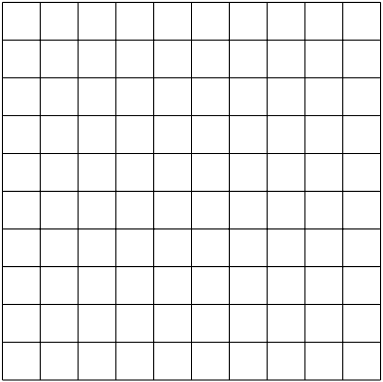

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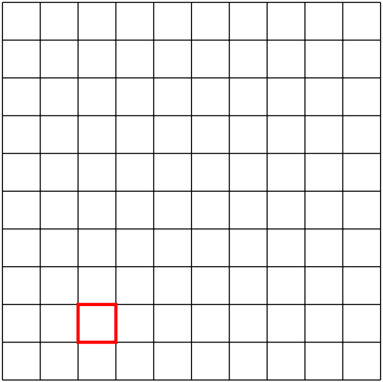

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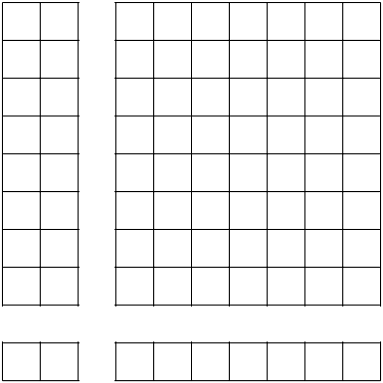

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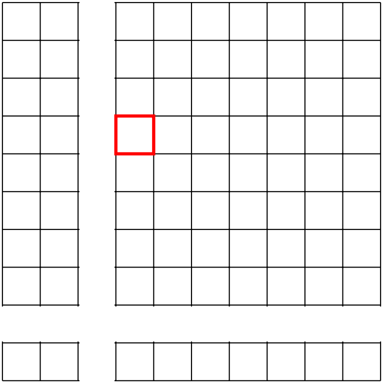

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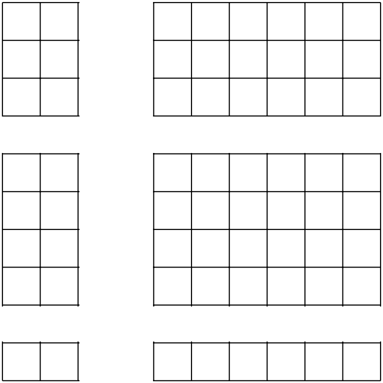

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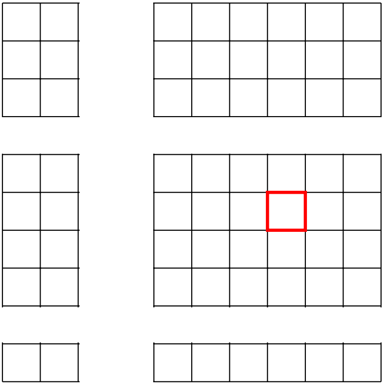

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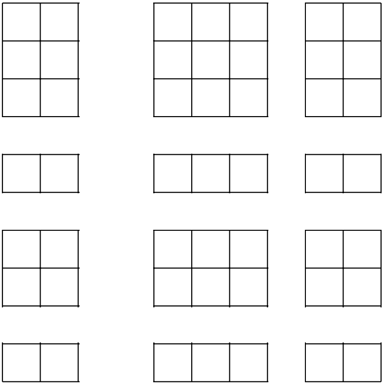
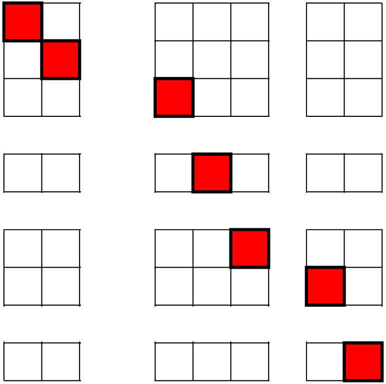


Lower bound for $\Gamma_{k,\ell}(\mathsf{Perm}_n)$

$$\Gamma_{k,\ell}(\mathsf{Perm}_n) = \dim\left\{\mathbf{x}^{\leq \ell}\,\partial^{=k}\mathsf{Perm}_n\right\}$$

$$\geq \#\left\{\begin{array}{l}\text{distinct leading monomials} \\ \text{in} \quad \mathbf{x}^{\leq \ell}\cdot\partial^{=k}(\mathsf{Perm}_n)\end{array}\right\}$$



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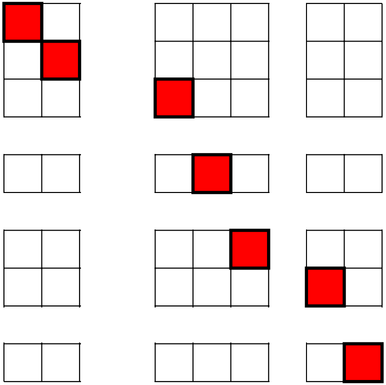
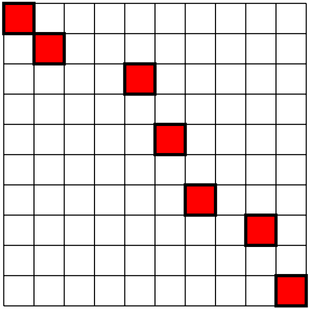
$$\begin{aligned}\Gamma_{k,\ell}(\mathsf{Perm}_n) &= \dim\left\{\mathbf{x}^{\leq \ell}\partial^{=k}\mathsf{Perm}_n\right\} \\ &\geq \#\left\{\begin{array}{l}\text{monomials of degree } \ell+n-k \\ \text{having an } (n-k) \textit{ inc. sequence}\end{array}\right\}\end{aligned}$$



Counting...



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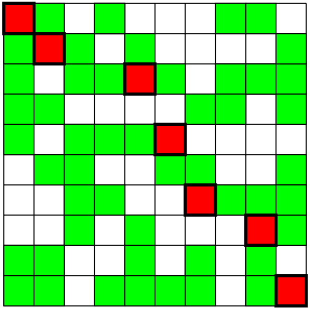

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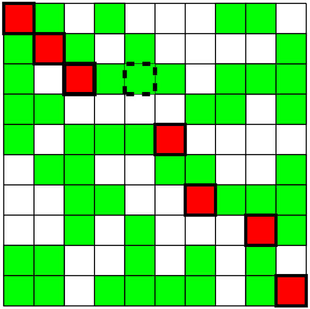

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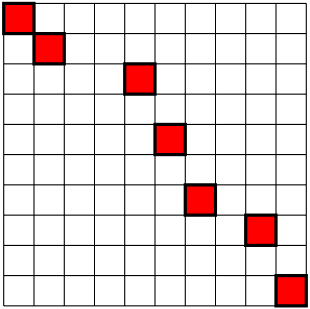

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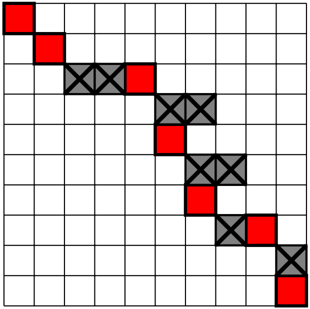

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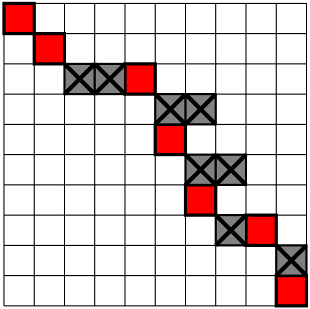

"Most" increasing sequences have about $2k$ forbidden elements.

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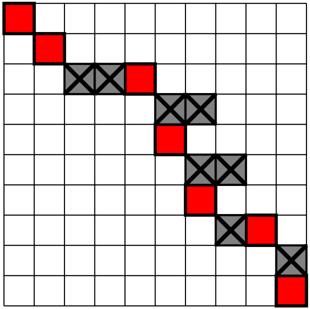

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$$\Gamma_{k,\ell}(\mathsf{Perm}_n) \quad \geq \quad \binom{n+k}{2k} \cdot \binom{\ell + n^2 - 2k}{n^2 - 2k}$$

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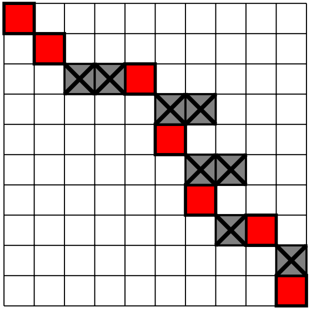

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$$\Gamma_{k,\ell}(\mathsf{Perm}_n) \quad \geq \quad \boxed{\text{Expression 2}}$$

Putting it together

$$C = \sum_{i=1}^{s} Q_{i1} \dots Q_{i\sqrt{n}} \qquad \deg(Q_{ij}) = \sqrt{n}$$

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$$\Gamma_{k,\ell}(\mathsf{Perm}_n) \geq \boxed{\text{Expression 2}}$$

Choosing ℓ and k carefully, $s = 2^{\Omega(\sqrt{n})}$.

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[Sturmfels90]: Det_n minors form a gröbner basis

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Question: But how much larger?

$$\Gamma_{k,\ell}(\mathrm{Det}_n) = \dim\left\{\mathbf{x}^{\leq \ell}\,\partial^{=k}\,\mathrm{Det}_n\right\}$$

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Any asymptotic improvement implies VP $\neq$ VNP!

Conjecture

VP *vs* VNP *would be resolved within the next 5 years.*

Let's get started...

Thank you!

Questions?